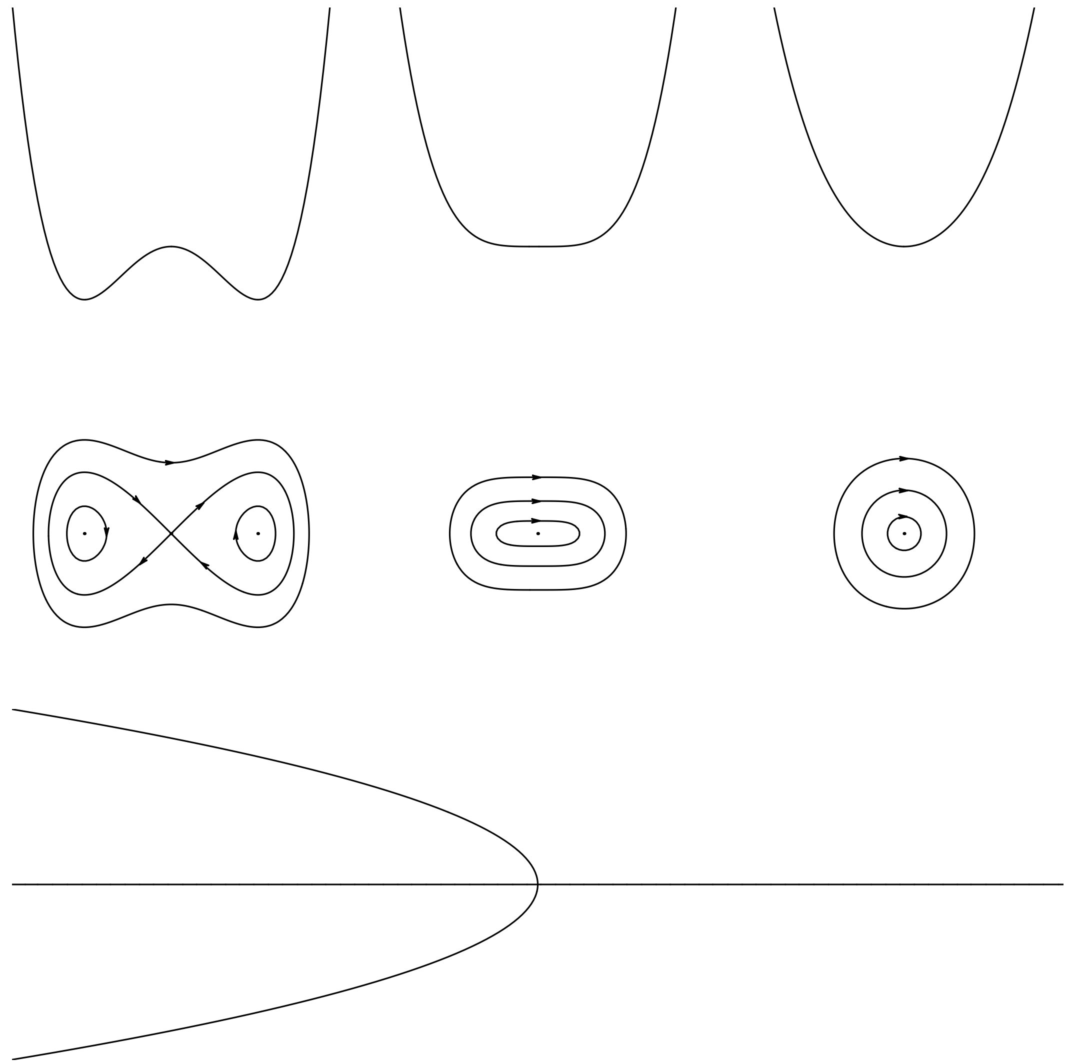

Fig. 2.1. The Hamiltonian pitchfork bifurcation.

The typical way in which a parabolic equilibrium bifurcates is the centre-saddle bifurcation. Here the Hamiltonian reads

$$H(x, y) \quad = \quad \frac{a}{2} y^2 \; + \; \frac{b}{6} x^3 \; + \; c\lambda x \tag{2.2}$$

where $a, b, c \in \mathbb{R}$ are nonzero constants. For instance, when $a = b = c = 1$ this leads to the phase portraits given in Fig. 2.2.

Note that this is a completely different unfolding of the parabolic equilibrium at the origin. A closer look at the phase portraits and in particular at the Hamiltonian function of the Hamiltonian pitchfork bifurcation reveals the symmetry $x \mapsto -x$. This suggests to add the non-symmetric term μx.

Exercise 2.8. Determine the bifurcation diagram of the family

$$H_{\lambda,\mu}(x, y) \quad = \quad \frac{1}{2} y^2 \; + \; \frac{1}{24} x^4 \; + \; \frac{\lambda}{2} x^2 \; + \; \mu x$$

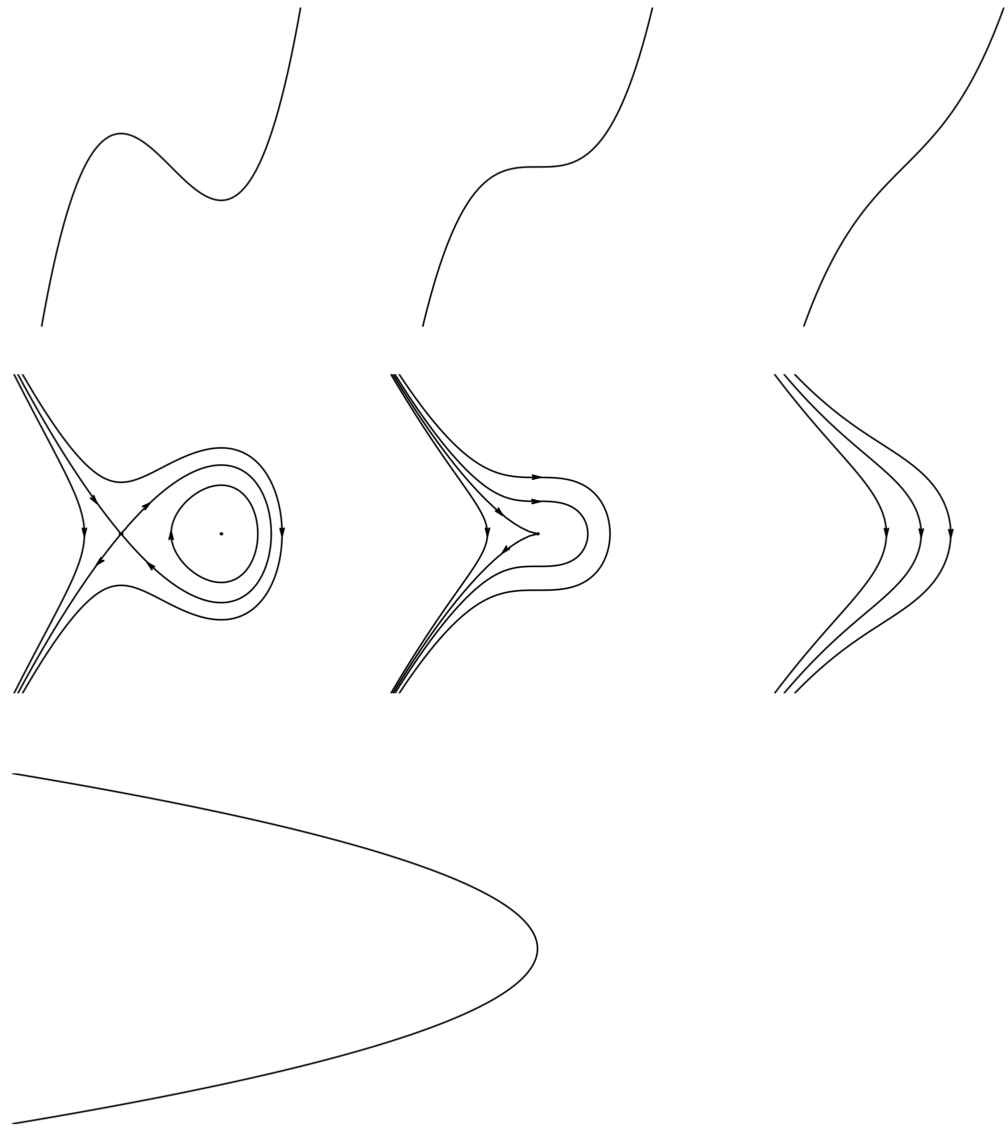

Fig. 2.2. The centre-saddle bifurcation.

of Hamiltonian systems. △

Singularity theory allows to prove that upon adding further "small" terms to $H_{\lambda,\mu}$ no additional phase portraits are generated. Up to equivalence (*i.e.* qualitatively) $H_{\lambda,\mu}$ contains all possible unfoldings of the anharmonic oscillator (2.1), one also speaks of a versal unfolding. Similarly, the centre-saddle bifurcation is a stable 1–parameter family.

Exercise 2.9. Analyse the family

$$H_\lambda(x,y) \;=\; \frac{1}{2}y^2 \;+\; \frac{1}{6}x^3 \;+\; \frac{\lambda}{2}x^2$$

of Hamiltonian systems. △

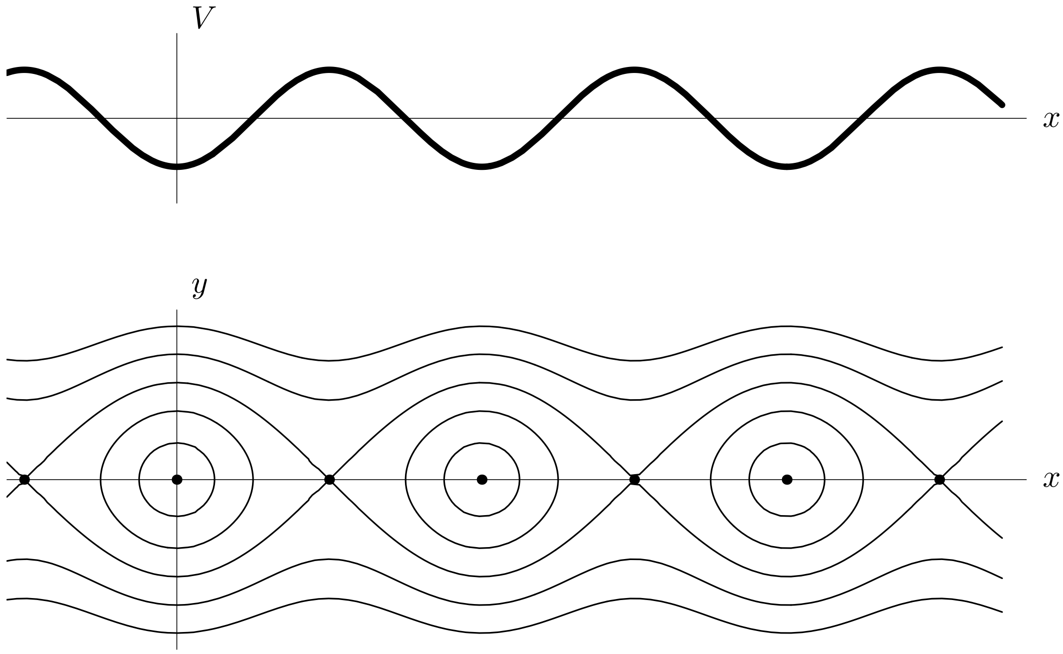

Fig. 2.3. The mathematical pendulum.

For any potential energy $V : S^1 \longrightarrow \mathbb{R}$ the Hamiltonian $H = T + V$ with kinetic energy $T = \frac{1}{2}y^2$ defines equations of motion

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -V'(x) \end{aligned}$$

on $S^1 \times \mathbb{R}$; the corresponding flow may be computed directly on the phase space or first on $\mathbb{R}^2$ and then projected mod 2π in the first component.

Exercise 2.11. Consider the Hamiltonian function $H(x, y) = \frac{1}{2}y^2 - \cos x$ of the pendulum. For $|z| < 1$ we consider the level set $H^{-1}(z)$. What is the amplitude of oscillation in this level? If $T(z)$ denotes the period of oscillation in this level, then give an explicit integral expression for this. Determine $\lim_{z\to -1} T(z)$ and $\lim_{z\to +1} T(z)$. △

Exercise 2.12. Analyse the dynamics of the rotating pendulum $\ddot{x} = M - \sin x$ in dependence of M. △

Exercise 2.13. Let $H(x, y) = \frac{1}{2}y^2 - V(x)$ be the Hamiltonian of a 1–dimensional particle with mass $m = 1$ moving in the potential V. Describe the time parametrisations of the trajectories $H = h$ in terms of the indefinite integral

$$\int \frac{\mathrm{d}x}{\sqrt{2(h - V(x))}} \ . \tag{2.3}$$

△

$$T_{(u,v,w)}S^2_r = \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{\perp}$$

and consists of all possible vectors $X_H(u,v,w)$ where H runs through all of $C^\infty(\mathbb{R}^3)$.

Example 3.5. Consider on S^2 (or, in fact, on $\mathbb{R}^3$) the Hamiltonian function

$$H(u,v,w) = \frac{au^2}{2} + \frac{bv^2}{2} + \frac{cw^2}{2} \tag{3.4}$$

where $0 < a \leq b \leq c$ are three real parameters. (This models part of the dynamics of a free rigid body with a fixed point, subject only to its own inertia.) The equations of motion

$$\begin{aligned} \dot{u} &= \{u,H\} = (b-c)vw \\ \dot{v} &= \{v,H\} = (c-a)uw \\ \dot{w} &= \{w,H\} = (a-b)uv \end{aligned}$$

are non-linear but can still be explicitly solved, using elliptic functions. Since the orbits coincide with the intersections $S^2 \cap \{H = h\}$ of the phase space with the energy level sets one can alternatively obtain the phase portrait intersecting the sphere $S^2 \subseteq \mathbb{R}^3$ with the ellipsoid $\{H = h\}$, see Fig. 3.1. □

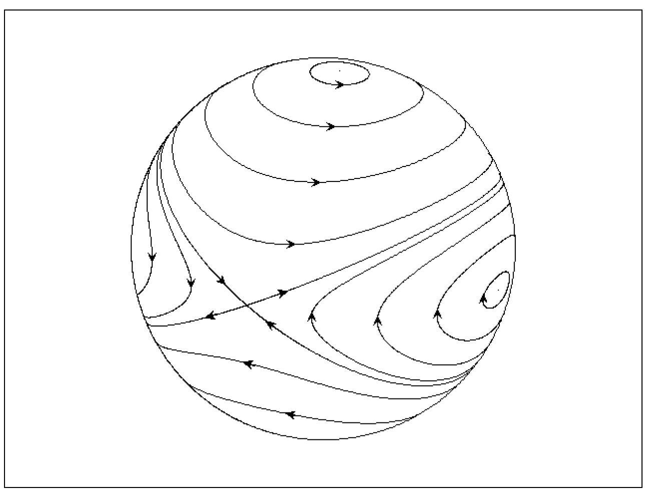

Fig. 3.1. Flow defined by the Hamiltonian (3.4).

Exercise 3.12. Analyse the dynamics defined by (3.4) on S^2 in the limiting cases $a \to b$ and $b \to c$. △