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This paper is based on two sources [1] [2], the numbers of the definitions, lemmas and theorems correspond to those in the sources for ease of reference.

Consider a smooth system ($f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1-smooth)

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

Recall that the corresponding flow $\varphi^t(x)$ is at least C^1 jointly in (t, x).

Let $\Gamma_0 \subset \mathbb{R}^n$ be a periodic orbit (cycle) of the dynamical system generated by (1), i.e. there exists $T > 0$ (the minimal period) such that for every $x_0 \in \Gamma_0$ we have $\varphi^T(x_0) = x_0$ and $\varphi^t(x_0) \neq x_0$ for $t \in (0, T)$.

Choose $x_0 \in \Gamma_0$ and define

$$\Sigma_0 = \{\xi \in \mathbb{R}^n : \langle f(x_0), \xi \rangle = 0\}$$

and introduce a cross-section

$$\Pi_{x_0} = \{x \in \mathbb{R}^n : x = x_0 + \xi, \xi \in \Sigma_0\}.$$

The orbit starting at x_0 (Γ_0) hits Π_{x_0} again after T units of time. Our next aim is to show that orbits of (1) starting at points on Π_{x_0} near x_0 also hit Π_{x_0} after approximately T units of time. Actually, we show that this is true for all orbits starting near x_0, either on or off Π_{x_0} (see Figure 1).

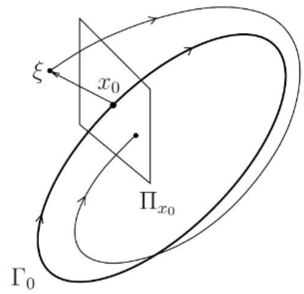

Figure 1: construction of the Poincare mapping

Lemma 3.11 (return to cross-section Π_{x_0}) There exists a C^1 map $\tau : \mathbb{R}^n \to \mathbb{R}, \xi \mapsto \tau(\xi)$, defined in a neighbourhood of $\xi = 0$ and such that
(i) $\tau(0) = T$;
(ii) $\varphi^{\tau(\xi)}(x_0 + \xi) \in \Pi_{x_0}$.

Proof. Define $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$F(t, \xi) = \left\langle f(x_0), \varphi^t(x_0 + \xi) - x_0 \right\rangle$$

and consider the equation

$$F(t, \xi) = 0$$

Since $\varphi^t(x)$ is at least C^1 jointly in (t, x), $F \in C^1$. We can compute the derivative over t of F,

$$F_t(t, \xi) = \left\langle f(x_0), \frac{\partial}{\partial t}\varphi^t(x_0 + \xi) \right\rangle = \langle f(x_0), f(\varphi^t(x_0 + \xi)) \rangle$$

Note that

$$F(T, 0) = \left\langle f(x_0), \varphi^T(x_0) - x_0 \right\rangle = \langle f(x_0), x_0 - x_0 \rangle = 0$$

while

$$F_t(T, 0) = \left\langle f(x_0), f(\varphi^T(x_0)) \right\rangle = \langle f(x_0), f(x_0) \rangle = \|f(x_0)\|^2 \neq 0$$

The Implicit Function Theorem now yields that there exists an open neighbourhood of $\xi = 0$ on which there exists a C^1 map $\tau : \mathbb{R}^n \to \mathbb{R}$ such that

$$\tau(0) = T, \quad F(\tau(\xi), \xi) = 0$$

Thus for any ξ in this neighbourhood,

$$\varphi^{\tau(\xi)}(x_0+\xi)-x_0\in\Sigma_0$$

and

$$x_0+\varphi^{\tau(\xi)}(x_0+\xi)-x_0=\varphi^{\tau(\xi)}(x_0+\xi)\in\Pi_{x_0}$$

$\square$

Definition 3.12 The map $\mathcal{P}:\Sigma_0\to\Sigma_0$, defined for $\xi\in\Sigma_0$ near $\xi=0$ by the formula

$$\mathcal{P}(\xi)=\varphi^{\tau(\xi)}(x_0+\xi)-x_0, \tag{2}$$

is called a Poincaré map of the periodic orbit Γ_0.

remark $\mathcal{P}$ is a (locally defined) map on the $(n-1)$-dimensional subspace Σ_0. Let $N_i\in\mathbb{R}^n, i=1,2,\dots,n-1$, be linearly independent vectors in Σ_0, so that

$$\langle N_i, f(x_0)\rangle=0$$

Then any $\xi\in\Sigma_0$ can be written as

$$\xi=\eta_1N_1+\eta_2N_2+\cdots+\eta_{n-1}N_{n-1}$$

We will denote this map as

$$N:\mathbb{R}^{n-1}\to\mathbb{R}^n,\quad \eta\mapsto N\xi$$

where N denotes the $n\times(n-1)$ matrix

$$\begin{pmatrix}N_1 & N_2 & \dots & N_{n-1}\end{pmatrix}$$

Since the columns of the matrix are linearly independent and span Σ_0 there exists an inverse matrix N^{-1} restricted to Σ_0. Given such coordinates η in Σ_0, the map $\mathcal{P}$ as defined in (2) is fully described by a (local) C^1-map

$$P:\mathbb{R}^{n-1}\to\mathbb{R}^{n-1},\quad \eta\mapsto P(\eta)$$

defined by

$$P(\eta)=N^{-1}\mathcal{P}(N\eta)$$

which is often also called the Poincaré map of Γ_0. Note that $\eta=0$ is a fixed point of this map: $P(0)=0$. The derivatives of $P(\eta)$ and $\mathcal{P}(\xi)$ are related by

$$\mathcal{P}_\xi(\xi)=NP_\eta(N^{-1}\xi)N^{-1}$$

The eigenvalues of the $(n-1)\times(n-1)$-matrix $P_\eta(0)$ are the eigenvalues of the linear map $\mathcal{P}_\xi(0)$. Indeed,

$$\mathcal{P}_\xi(0)v=\lambda v=NP_\eta(0)N^{-1}v$$

$$\lambda N^{-1}v=P_\eta(0)N^{-1}v$$

and conversely

$$P_\eta(0)v=\lambda v=N^{-1}\mathcal{P}_\xi(0)Nv$$

$$\lambda Nv=\mathcal{P}_\xi(0)Nv$$

So in particular $\mathcal{P}_\xi(0)$ restricted to Σ_0 has $(n-1)$ eigenvalues.

Restricting the neigbourhood of Lemma 1 to Σ_0 shows that the Poincaré map is well defined and since τ is C^1 the Poincaré map is smooth.

Theorem 3.13
(i) If all $(n-1)$ eigenvalues of the linearization $\mathcal{P}_\xi(0)$ of the Poincaré map $\mathcal{P}$ at $\xi=0$ satisfy $|\lambda|<1$, then Γ_0 is asymptotically stable.
(ii) If $|\lambda|>1$ for some eigenvalue λ of the linearization of the Poincaré map $\mathcal{P}$, then Γ_0 is unstable.

Proof. (i) By theorem 2.7 (see stability of linear maps section at the end of this document) introduce an equivalent norm $\|\cdot\|_1$ in $\mathbb{R}^n$ in which $\mathcal{P}_\xi(0)$ is a linear contraction on Σ_0. By theorem 3.1 there exists $\delta_0 > 0$ such that for all $\xi \in \Sigma_0$ with $\|\xi\|_1 \leq \delta_0$ the inequality

$$\|\mathcal{P}(\xi)\|_1 \leq \rho_1 \|\xi\|_1 \tag{3}$$

holds with some $\rho_1 < 1$. For any $\delta \leq \delta_0$, construct a neighbourhood U_δ of Γ_0 as follows. Take the ball in Σ_0

$$B_\delta = \{\xi \in \Sigma_0 : \|\xi\|_1 \leq \delta\}$$

and consider all orbits of (1) starting at $x_0 + \xi$ with $\xi \in B_\delta$. Any such orbit returns back to Π_0 after $\tau(\xi)$ units of time. Define now $U_\delta \subset \mathbb{R}^n$ as the union of all such orbit segments (see figure 2), i.e.,

$$U_\delta = \left\{x \in \mathbb{R}^n : x = \varphi^t\left(x_0 + \xi\right), \xi \in B_\delta, 0 \leq t \leq \tau(\xi)\right\}$$

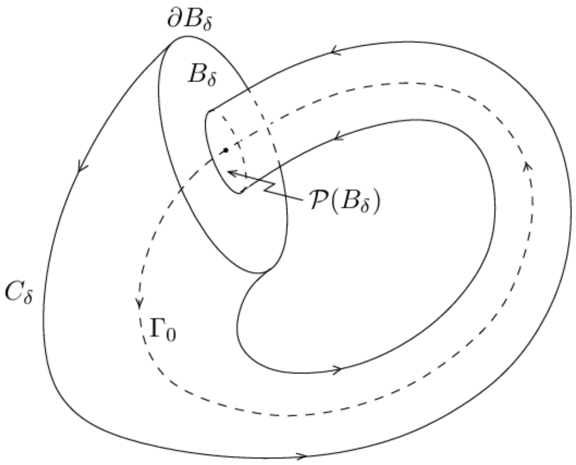

Figure 2: construction of U_δ

The set U_δ is a closed tubular neighbourhood of Γ_0 that shrinks to Γ_0 as $\delta \to 0$.
Indeed, because $\varphi^t(x)$ is C^1 in (t, x) it is Lipschitz continuous over the closed interval $[0, 2T]$. Thus for some $C \geq 1$,

$$\|\varphi^t(x_0) - \varphi^t(x_0 + \xi)\| \leq C\|(t, x_0) - (t, x_0 + \xi)\| = C\|\xi\| \tag{4}$$

This holds for all $t \in [0, 2T]$ (this includes the whole cycle for small enough δ since $\tau(\xi) \to T$ as $\delta \to 0$) and $\xi \in B_\delta$, thus U_δ shrinks to Γ_0 as $\delta \to 0$.

Since $\mathcal{P}(B_\delta)$ is located strictly in B_δ, U_δ is a trapping region, i.e. any orbit starting in U_δ remains in it for all $t \geq 0$.
Indeed, the boundary ∂U_δ of U_δ consists of a cylinder C_δ, which is formed by translations of all points of ∂B_δ by the flow until they return to Σ_0, and a set D_δ defined by

$$D_\delta = \operatorname{Int} B_\delta \setminus \operatorname{Int} \mathcal{P}(B_\delta),$$

which is an annulus in Σ_0 between B_δ and $\mathcal{P}(B_\delta)$. Provided δ is sufficiently small, since $f(x)$ is smooth and $f(x_0)$ is transverse to D_δ, all orbits of the ODE that start in D_δ cross it transversally and then enter U_δ. This implies that any orbit starting in U_δ cannot leave U_δ for $t \geq 0$, if one takes into account that the cylinder C_δ is positively invariant with respect to the system flow.

Consider now any small open neighbourhood U of Γ_0. Making δ sufficiently small, we can guarantee that $U_\delta \subset U$. Since U_δ is a trapping region, this implies Lyapunov stability of Γ_0.

By induction, it follows from (3) that

$$\left\|\mathcal{P}^k(\xi)\right\|_1 \leq \rho_1^k \|\xi\|_1, \quad k = 1, 2, 3, \cdots.$$

Let $\xi_k = \mathcal{P}^k(\xi)$, then since $\rho_1 < 1$, $\|\xi_k\|_1 \to 0$ as $k \to \infty$. This means that the forward half of any orbit starting in U_δ can be divided into finite segments, whose end-points $x_0 + \xi_k$ where $\xi_k \in \Sigma_0$, form a convergent sequence with $\{\xi_k\} \to 0$. Using (4) we see that these segments converge to Γ_0 since ξ_k converges to 0. We conclude that $\operatorname{dist}\left(\varphi^t\left(x_0+\xi\right), \Gamma_0\right) \to 0$ as $t \to +\infty$, i.e. Γ_0 is asymptotically stable.

(ii) This part also follows from Theorem 3.1, since instability for the Poincaré map $\mathcal{P}$ immediately implies instability of Γ_0 with respect to the flow generated by the ODE. $\square$

There is actually a stronger sense of stability near Γ_0 that follows from (3),

Definition 3.14 *A cycle Γ_0 through x_0 is called exponentially orbitally stable with asymptotic phase if there exist $c>0, K>1$, and $t_0=t_0(x) \in [0,T)$ such that*

$$\left\|\varphi^t(x)-\varphi^{t-t_0}\left(x_0\right)\right\| \leq K \mathrm{e}^{-ct}, \quad t \geq 0$$

for all x with sufficiently small $\operatorname{dist}\left(x, \Gamma_0\right)$.

Theorem 3.15 *If all $(n-1)$ eigenvalues of the linearization $\mathcal{P}_\xi(0)$ of the Poincaré map $\mathcal{P}$ at $\xi=0$ satisfy $|\lambda|<1$, then Γ_0 is exponentially orbitally stable with asymptotic phase.*

Proof. Consider first $x=x_0+\xi, \quad \xi \in \Sigma_0$, and define for a given ξ :

$$\begin{aligned}
\xi_0 &= \xi, \\
\xi_k &= \mathcal{P}\left(\xi_{k-1}\right), \\
\tau_k &= \tau\left(\xi_{k-1}\right)+\tau_{k-1}, \tau_0=0
\end{aligned}$$

for $k=1,2,\ldots$. Using (3) we know that (we denote the norm $\|\cdot\|_1$ by $\|\cdot\|$) there exists $\delta>0$ such that for all ξ with $\|\xi\| \leq \delta$ the estimate,

$$\|\xi_k\| = \|\mathcal{P}^k(\xi)\| \leq \rho_1^k \|\xi\| \leq e^{-\alpha k}\|\xi\|$$

holds. The last inequality follows from the fact that $\rho_1<1$ and,

$$\rho_1^k = e^{k \ln \rho_1} \leq e^{-\alpha k}$$

for some $-\alpha \geq \ln \rho_1$, we choose $-\alpha<0$.
Since $\tau \in C^1$, it is Lipschitz continuous on $[0, \delta]$ and we can derive the estimate:

$$\left|\tau\left(\xi_{k-1}\right)-T\right| = \left|\tau\left(\xi_{k-1}\right)-\tau(0)\right| \leq C\left\|\xi_{k-1}\right\| \leq C \mathrm{e}^{-\alpha(k-1)}\left\|\xi_0\right\|$$

This implies

$$\left|\left(\tau_k-kT\right)-\left(\tau_{k-1}-(k-1)T\right)\right| = \left|\tau\left(\xi_{k-1}\right)-T\right| \leq C \mathrm{e}^{-\alpha(k-1)}\left\|\xi_0\right\| .$$

Thus, $\theta_k=\tau_k-kT$ is a Cauchy sequence, so it has a limit that we denote by t_0. By iteratively applying the previous inequality we have

$$\left|\tau_{k+m}-(k+m)T-\left(\tau_k-kT\right)\right| \leq C\left\|\xi_0\right\| \sum_{j=0}^{m-1} \mathrm{e}^{-\alpha(k+j)} \leq C\left\|\xi_0\right\| \mathrm{e}^{-\alpha k} \sum_{j=0}^{m-1} \mathrm{e}^{-\alpha j} \leq C\left\|\xi_0\right\| \frac{\mathrm{e}^{-\alpha k}}{1-\mathrm{e}^{-\alpha}}$$

where the last inequality follows from the geometric series and the fact that $e^{-\alpha}<1$. Taking the limit $m \to +\infty$, we find

$$\left|\tau_k-kT-t_0\right| \leq C\left\|\xi_0\right\| \frac{\mathrm{e}^{-\alpha k}}{1-\mathrm{e}^{-\alpha}}$$

We now apply the Lipschitz continuity of $\varphi^t(x)$ given by (4) to obtain,

$$\left\|\varphi^{t+\tau_k}\left(x_0+\xi_0\right)-\varphi^t\left(x_0\right)\right\| = \left\|\varphi^t\left(x_0+\xi_k\right)-\varphi^t\left(x_0\right)\right\| \leq C_1\left\|\xi_k\right\| \leq C_1 \mathrm{e}^{-\alpha k}\left\|\xi_0\right\|$$

for $0 \leq t \leq T$. Likewise

$$\begin{aligned}
\left\|\varphi^{t+\tau_k}\left(x_0+\xi_0\right)-\varphi^{t+kT+t_0}\left(x_0+\xi_0\right)\right\| &= \left\|\varphi^t\left(x_0+\xi_k\right)-\varphi^t\left(\varphi^{kT+t_0-\tau_k}\left(x_0+\xi_k\right)\right)\right\| \\
&= \left|\tau_k-kT-t_0\right| \\
&\leq C\left\|\xi_0\right\| \frac{\mathrm{e}^{-\alpha k}}{1-\mathrm{e}^{-\alpha}} \\
&= C_2 \mathrm{e}^{-\alpha k}\left\|\xi_0\right\|,
\end{aligned}$$

for $0 \leq t \leq T$. Combining the last two inequalities and using the periodicity of $\varphi^t(x_0)$, we find

$$\left\|\varphi^{t+t_0}\left(x_0+\xi_0\right)-\varphi^t\left(x_0\right)\right\| \leq \left(C_1+C_2\right) \mathrm{e}^{-\alpha k}\left\|\xi_0\right\|$$

for $kT \leq t \leq (k+1)T$. Now take any $x \in \mathbb{R}^n$ near Γ_0. If this point does not belong to Π_0, consider the first intersection of the forward half-orbit starting at x with Π_0 and represent it as $x_0+\xi_0$. Apply then the above given proof. $\square$

Note that in the following section $\frac{\partial}{\partial x}$ denotes the total derivative of a function w.r.t. the variable x, this is also denoted using subscript x.

Lemma 3.6 The matrix

$$Y(t) = \left.\frac{\partial \varphi^t(x)}{\partial x}\right|_{x=x_0}$$

satisfies the linear differential equation

$$\dot{Y} = f_x\left(\varphi^t\left(x_0\right)\right) Y$$

and the initial condition $Y(0) = I_n$.

Proof. Let $x\left(t, x_0 + hv\right) = \varphi^t\left(x_0 + hv\right)$. Note that $Y(t)$ is the total derivative of φ^t w.r.t. x, so any directional derivative can be written as $Y(t)v$ (where v is the direction). Thus, for any $v \in \mathbb{R}^n$:

$$[Y(t)]v = \lim_{h\to 0} \frac{1}{h}\left[x\left(t, x_0 + hv\right) - x\left(t, x_0\right)\right].$$

Now

$$\begin{aligned}[\dot{Y}(t)]v = \frac{d}{dt}[Y(t)]v &= \lim_{h\to 0} \frac{1}{h}\left[\dot{x}\left(t, x_0 + hv\right) - \dot{x}\left(t, x_0\right)\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[f\left(x\left(t, x_0 + hv\right)\right) - f\left(x\left(t, x_0\right)\right)\right]\end{aligned}$$

Note that this is the directional derivative of $f(\varphi^t(x))$ in the direction v in the point x_0. Thus,

$$\begin{aligned}[\dot{Y}(t)]v &= \left[\left.\frac{\partial}{\partial x} f\left(\varphi^t\left(x\right)\right)\right|_{x=x_0}\right] v \\ &= \left[\left. f_x\left(\varphi^t\left(x\right)\right) \cdot \varphi_x^t(x)\right|_{x=x_0}\right] v \\ &= f_x\left(\varphi^t\left(x_0\right)\right) Y(t)v\end{aligned}$$

for any $v \in \mathbb{R}^n$, so $\dot{Y} = f_x\left(\varphi^t\left(x_0\right)\right) Y$. Since $\varphi^0(x) = x, Y(0) = I_n$. ☐

Note that $Y(T)$ is also dependent on the initial point x_0, while this is not explicitly written.

Lemma 3.7 Let $y_0 = f\left(x_0\right)$ and $y_1 = f\left(\varphi^{t_1}\left(x_0\right)\right)$. Then $y_1 = Y\left(t_1\right) y_0$.

Proof. Since $x(t) = \varphi^t\left(x_0\right)$ is a solution to (1), we have

$$\frac{d}{dt}\varphi^t\left(x_0\right) = f\left(\varphi^t\left(x_0\right)\right)$$

Differentiating this equation with respect to t we find

$$\frac{d}{dt}\left(\frac{d}{dt}\varphi^t\left(x_0\right)\right) = f_x\left(\varphi^t\left(x_0\right)\right)\frac{d}{dt}\varphi^t\left(x_0\right)$$

so

$$y(t) = \frac{d}{dt}\varphi^t\left(x_0\right) = f\left(\varphi^t\left(x_0\right)\right)$$

is a solution to the linearized problem

$$\dot{y} = f_x\left(\varphi^t\left(x_0\right)\right) y, \quad y \in \mathbb{R}^n$$

with the initial condition $y(0) = f\left(x_0\right) = y_0$. Since any such solution has the form $y(t) = Y(t)y_0$ (Y(t) is the fundamental matrix solution by lemma 3.6), we get

$$y_1 = y\left(t_1\right) = Y\left(t_1\right) y_0$$

☐

Theorem 3.8 For any $x_0 \in \Gamma_0, f\left(x_0\right)$ is an eigenvector of $Y(T)$ corresponding to eigenvalue 1.

Proof. By Lemma 3.7, $f\left(\varphi^T\left(x_0\right)\right) = Y(T)f\left(x_0\right)$. The periodicity now yields $f\left(x_0\right) = Y(T)f\left(x_0\right)$. ☐

Definition 3.9 $Y(T)$ is called the monodromy matrix. Its eigenvalues are called the (characteristic or) Floquet multipliers. The multiplier 1 is called trivial, while all others are called nontrivial multipliers.

Definition 3.10 A cycle Γ_0 of (1) is called simple if $\lambda = 1$ is a simple eigenvalue of $Y(T)$.

We're now going to establish a relationship between the eigenvalues of the linear part $\mathcal{P}_\xi(0)$ of the Poincaré mapping and the eigenvalues of the monodromy matrix $Y(T)$.

Lemma 3.16 (i) The linearization around $\xi = 0$ of $\mathcal{P}(\xi)$ is the restriction to Σ_0 of the linear map

$$\xi \mapsto \langle \tau_\xi(0), \xi \rangle \, f(x_0) + Y(T)\xi \tag{5}$$

(ii) Take a point in $\mathbb{R}^n$. $f(x_0)$ and Σ_0 span $\mathbb{R}^n$ so we can write this point as $cf(x_0) + \xi$ for $c \in \mathbb{R}$ and $\xi \in \Sigma_0$. Denote this point by its span $\{f(x_0)\}$ and Σ_0 component as (c, ξ). Then $Y(T)$ maps

$$(c, \xi) \mapsto (c - \langle \tau_\xi(0), \xi \rangle, \mathcal{P}_\xi(0)\xi)$$

.

Proof. (i) By Lemma 3.11 the map $\xi \mapsto \tau(\xi)$ is defined and differentiable in a neighbourhood of the origin in $\mathbb{R}^n$. Since $\varphi^t(x)$ is differentiable in both t, x the same is true for the Poincaré map $\xi \mapsto \mathcal{P}(\xi) = \varphi^{\tau(\xi)}(x_0 + \xi) - x_0$. The derivative of the Poincaré map is determined using the chain rule. To make clear the steps of taking the derivative we define the following functions,

$$\varphi(t, x) = \varphi^t(x), \quad g(\xi) = x_0 + \xi$$

$$\mathcal{P}_\xi(\xi) = \frac{d}{d\xi}(\varphi(\tau(\xi), g(\xi)) - x_0) = \frac{\partial \varphi}{\partial t}\frac{d\tau}{d\xi} + \frac{\partial \varphi}{\partial x}\frac{dg}{d\xi} = \frac{\partial \varphi}{\partial t}\frac{d\tau}{d\xi} + \frac{\partial \varphi}{\partial x}$$

Since φ is a flow of the system (1) $\frac{\partial \varphi}{\partial t} = f(\varphi)$. Substituting this and writing the arguments of the functions gives,

$$\mathcal{P}_\xi(\xi) = f(\varphi(\tau(\xi), g(\xi)))\tau_\xi(\xi) + \varphi_x(\tau(\xi), g(\xi))$$
$$\mathcal{P}_\xi(0) = f(\varphi(T, x_0))\tau_\xi(0) + \varphi_x(T, x_0)$$

Since $Y(T) = \varphi_x^T(x_0)$ and $\varphi(T, x_0) = x_0$,

$$\mathcal{P}_\xi(0) = f(x_0)\tau_\xi(0) + Y(T)$$

Thus the linearization around $\xi = 0$ is given by (5). Next we simply restrict to Σ_0.
(ii) Since $Y(T)f(x_0) = f(x_0)$, the point with coordinates (c, ξ) is mapped to $cf(x_0) + Y(T)\xi$. According to part (i) we may write

$$Y(T)\xi = \mathcal{P}_\xi(0)\xi - \langle \tau_\xi(0), \xi \rangle \, f(x_0)$$

$\mathcal{P}$ maps points on Σ_0 to Σ_0. Since Σ_0 is an affine vector subspace of $\mathbb{R}^n$, the derivative of $\mathcal{P}$ also maps to Σ_0,

$$\mathcal{P}_\xi(0)\xi \in \Sigma_0$$

So the image point has coordinates $(c - \langle \tau_\xi(0), \xi \rangle, \mathcal{P}_\xi(0)\xi)$. □

Theorem 3.17 (i) $\lambda \neq 1$ is an eigenvalue of $P_\xi(0)$ if and only if λ is an eigenvalue of $Y(T)$.
(ii) $\lambda = 1$ is an eigenvalue of $P_\xi(0)$ if and only if the eigenvalue 1 of $Y(T)$ has multiplicity bigger than one.

Proof. (i) If $Y(T)\eta = \lambda\eta$ and η has coordinates (c, ξ), then $\mathcal{P}_\xi(0)\xi = \lambda\xi$ because of Lemma 3.16 (ii) and the fact that span $\{f(x_0)\}$ and Σ_0 are linearly independent. If $\lambda \neq 1$ then $\xi \neq 0$ since the eigenvector corresponding to $\xi = 0$ of $Y(T)$ is $f(x_0)$ which has eigenvalue 1. On the other hand, if $\mathcal{P}_\xi(0)\xi = \lambda\xi$ and $\lambda \neq 1$, then η given by

$$\eta = \frac{1}{1-\lambda}\langle \tau_\xi(0), \xi \rangle \, f(x_0) + \xi$$

is such that

$$\begin{aligned} Y(T)\eta &= \frac{1}{1-\lambda}\langle \tau_\xi(0), \xi \rangle \, Y(T)f(x_0) + \mathcal{P}_\xi(0)\xi - \langle \tau_\xi(0), \xi \rangle \, f(x_0) \\ &= \lambda\xi + (\frac{1}{1-\lambda} - 1)\langle \tau_\xi(0), \xi \rangle \, f(x_0) \\ &= \lambda\xi + (\frac{\lambda}{1-\lambda})\langle \tau_\xi(0), \xi \rangle \, f(x_0) = \lambda\eta \end{aligned}$$

In the above calculation we used (5) and theorem 3.8.
(ii) Suppose first that η is not a multiple of $f(x_0)$. We can distinguish two cases,

$$Y(T)\eta = \eta$$

and

$$Y(T)\eta - \eta = f(x_0)$$

For case one, the Σ_0-component ξ of η is nonzero. And, by Lemma 3.16 (ii), $\mathcal{P}_\xi(0)\xi = \xi$, so 1 is an eigenvalue of $\mathcal{P}_\xi(0)$.
For case two write $\eta = cf(x) + \xi$. Then it follows from Lemma 3.16 (ii) that

$$(c - \langle \tau_\xi(0), \xi \rangle) f(x_0) + \mathcal{P}_\xi(0)\xi - cf(x_0) - \xi = f(x_0)$$

$$\mathcal{P}_\xi(0)\xi - \xi = 0$$

so $\mathcal{P}_\xi(0)\xi = \xi$ where ξ is the Σ_0-component of η which is nonzero.

If, conversely, $\mathcal{P}_\xi(0)\xi = \xi$ we distinguish the case where $\langle \tau_\xi(0), \xi \rangle \neq 0$ from the case where $\langle \tau_\xi(0), \xi \rangle = 0$. In the latter case it follows from (5) that $Y(T)\xi = \mathcal{P}_\xi(0)\xi = \xi$, so 1 is an eigenvalue of $Y(T)$ and ξ is not a multiple of $f(x_0)$ since the image of $\mathcal{P}_\xi(0)$ is Σ_0. Thus the eigenvalue 1 has multiplicity bigger than one. In the former case, we find that the normalized vector

$$\zeta = -\frac{1}{\langle \tau_\xi(0), \xi \rangle}\xi$$

satisfies

$$\begin{aligned} Y(T)\zeta - \zeta &= -\langle \tau_\xi(0), -\frac{1}{\langle \tau_\xi(0), \xi \rangle}\xi \rangle f(x_0) + \mathcal{P}_\xi(0)\left(-\frac{1}{\langle \tau_\xi(0), \xi \rangle}\xi\right) - \zeta \\ &= \zeta + f(x_0) - \zeta \\ &= f(x_0) \end{aligned}$$

showing that, corresponding to the eigenvalue $1, Y(T)$ has a higher-than-one dimensional generalized eigenspace and thus that the eigenvalue 1 has a multiplicity bigger than 1. □

Theorem 3.17 implies that

$$\det(\lambda I_n - Y(T)) = (\lambda - 1)\det(\lambda I_{n-1} - P_\eta(0))$$

where P_ξ is defined in a remark immediately after Definition 3.12. Furthermore, by combining Theorems 3.13, 3.15, and 3.17 we arrive at the following summarising result.

Theorem 3.18 If all nontrivial Floquet multipliers of a simple cycle have modulus less than one, then the cycle is exponentially orbitally stable with asymptotic phase. If some multiplier lies outside the unit circle, the cycle is unstable.

stability of linear maps

Definition 2.4 The spectral radius of a linear map A is defined by

$$r(A) = \sup_{\lambda \in \sigma(A)} |\lambda|$$

The relation between these quantities is specified by Gelfand's formula which we will state without proof,

$$r(A) = \lim_{k \to \infty} \left\| A^k \right\|^{1/k} = \inf_{k \geq 1} \left\| A^k \right\|^{1/k} \tag{6}$$

This shows that the eigenvalues of A yield information about the growth or decay of the time-series obtained by iterating A.
Theorem 2.7 Let $\rho > r(A)$. There exists an equivalent norm $\|\cdot\|_1$ on $\mathbb{R}^n$ such that $\|A\|_1 \leq \rho$.

Proof. Define $\|\cdot\|_1$ for $x \in \mathbb{R}^n$ by the formula:

$$\|x\|_1 = \sum_{k=0}^{\infty} \rho^{-k} \left\|A^k x\right\|$$

Formula (6) implies that this series converges. Indeed, for k sufficiently large and some $q < 1$,

$$\left\|A^k\right\|^{1/k} \leq \rho q$$

$$\rho^{-1} \left\|A^k\right\|^{1/k} \leq q$$

and hence

$$\rho^{-k} \left\|A^k x\right\| \leq \rho^{-k} \left\|A^k\right\| \|x\| \leq \|x\| q^k$$

$$\sum_{k=0}^{\infty} \rho^{-k} \left\|A^k x\right\| \leq \|x\| \sum_{k=0}^{\infty} q^k$$

Since $q < 1$ this implies that the sum converges. Clearly, $\|x\|_1 \geq 0$ for all $x \in \mathbb{R}^n$ and $\|x\|_1 = 0$ if and only if $\left\|A^k x\right\| = 0$ if and only if $x = 0$. Likewise the property $\|\alpha x\|_1 = |\alpha| \|x\|_1$ holds,

$$\|\alpha x\|_1 = \sum_{k=0}^{\infty} \rho^{-k} \left\|A^k \alpha x\right\| = |\alpha| \sum_{k=0}^{\infty} \rho^{-k} \left\|A^k x\right\| = |\alpha| \|x\|_1$$

and $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ also holds,

$$\|x + y\|_1 = \sum_{k=0}^{\infty} \rho^{-k} \left\|A^k x + A^k y\right\| \leq \sum_{k=0}^{\infty} \rho^{-k} (\left\|A^k x\right\| + \left\|A^k y\right\|) = \|x\|_1 + \|y\|_1$$

So $\|\cdot\|_1$ is a norm on $\mathbb{R}^n$ and since $\mathbb{R}^n$ is finite dimensional $\|\cdot\|_1$ is equivalent to $\|\cdot\|$. Now, for $x \in \mathbb{R}^n$,

$$\|Ax\|_1 = \sum_{k=0}^{\infty} \rho^{-k} \left\|A^{k+1} x\right\| = \rho \sum_{k=-1}^{\infty} \rho^{-(k+1)} \left\|A^{k+1} x\right\| - \rho \|A^0 x\| = \rho \left(\|x\|_1 - \|x\|\right)$$

so that

$$\|Ax\|_1 \leq \rho \|x\|_1, \quad x \in \mathbb{R}^n$$

□

Theorem 3.1 (Principle of Linearized Stability for Maps) Consider a C^1-map

$$x \mapsto g(x), \quad x \in \mathbb{R}^n$$

with $g(0) = 0$. Let $A = g_x(0)$.
(i) If $r(A) < 1$ then the fixed point $x = 0$ is asymptotically stable.
(ii) If $r(A) > 1$ then the fixed point $x = 0$ is unstable.

Proof. We will only prove part (i), for the proof of part (2) see [2].
(i) Take any ρ satisfying $r(A) < \rho < 1$. By theorem 2.7, there is a norm $\|\cdot\|_1$, which is equivalent to $\|\cdot\|$ and for which

$$\|Ax\|_1 \leq \rho \|x\|_1, \quad x \in \mathbb{R}^n$$

Since g is a C^1-map, for any small $\varepsilon > 0$, there is $\delta > 0$, such that

$$\|g(x) - Ax\|_1 \leq \varepsilon \|x\|_1$$

when $\|x\|_1 \leq \delta$. Then, for all such x,

$$\|g(x)\|_1 = \|Ax + g(x) - Ax\|_1 \leq \|Ax\|_1 + \|g(x) - Ax\|_1 \leq (\rho + \varepsilon) \|x\|_1$$

Since $\rho < 1$ and $\varepsilon > 0$ is arbitrarily small, we can achieve that $\rho_1 = \rho + \varepsilon < 1$, which implies that g maps the ball

$$\bar{B}_\delta = \{x \in \mathbb{R}^n : \|x\|_1 \leq \delta\}$$

into itself for all sufficiently small $\delta > 0$, so the fixed point $x = 0$ is stable. By induction:

$$\left\|g^k(x)\right\|_1 \leq \rho_1^k \|x\|_1$$

showing that $g^k(x) \to 0$ as $k \to +\infty$ for any x with $\|x\|_1 \leq \delta$. Therefore, the fixed point $x = 0$ is asymptotically stable. □

bibliography

References

[1] Yuri A. Kuznetsov, Odo Diekmann, and W. -J. Beyn. *Chapter 2 linear maps and odes*. Dec. 2011. URL: https://webspace.science.uu.nl/~kouzn101/NLDV/Lect2_3.pdf.

[2] Yuri A. Kuznetsov, Odo Diekmann, and W. -J. Beyn. *Chapter 3 Local behavior of nonlinear systems*. Dec. 2011. URL: https://webspace.science.uu.nl/~kouzn101/NLDV/Lect4_5.pdf.