Uniqueness of the limit cycle near BT bifurcation

Michiel Kaptijn

January 2025

Introduction

In this piece we will analyze the following two parameter planar system:

$$\begin{cases} \dot{\xi_1} = \xi_2, \\ \dot{\xi_2} = \beta_1 + \beta_2\xi_1 + \xi_1^2 - \xi_1\xi_2. \end{cases} \tag{1}$$

This is called the Bagdanov-Takens normal form. We will first analyse the bifurcation diagram from this system. In here we will discover that for certain values of β_1 and β_2 this system will have a limit cycle. We will then move on to proving that this limit cycle is unique.

Equilibria and Bifurcation Diagram

When analysing system (1) the first thing to look at will be any present equilibria. Looking at the first equation we can clearly see that any equilibria must satisfy $\xi_2 = 0$. Using this looking at the second equation we find that any equilibria must satisfy:

$$\beta_1 + \beta_2\xi_1 + \xi_1^2 = 0. \tag{2}$$

Since we are only interested in real solutions, this means that depending on the values of β_1, β_2 we will have between 0 and 2 equilibria. Looking at the discriminant of equation (2) we can highlight the following curve:

$$T = \{(\beta_1, \beta_2) \in \mathbb{R}^2 : \beta_2^2 - 4\beta_1 = 0\},$$

which constist of all pairs (β_1, β_2) for which system (1) only has one equilibrium. Suppose $\beta_2 \neq 0$. We can see when $\beta_2^2 > 4\beta_1$ then we are on the left side of the curve and system (1) will have two equilibria. We will call these equilibria $E_{1,2}$ which are given by:

$$E_{1,2} = (\xi_{1,2}^0, 0) = \left(\frac{-\beta_2 \mp \sqrt{\beta_2^2 - 4\beta_1}}{2}, 0 \right).$$

If $\beta_2^2 < 4\beta_1$ then we will be situated on the right side of T and system (1) will have no equilibria. So if we cross T from right to left system (1) goes from having no equilibria to having two. This indicates that a fold bifurcation occurs on this curve. For $\beta_2 = 0$ this fold bifurcation will be degenerate so we will not consider this case. We are now interested in finding out the stability of these equilibria. From the fold bifurcation we know we will get a node (which will be E_1) and a saddle (which will be E_2). To confirm this we compute the Jacobi-matrix: J:

$$J(\xi_1, \xi_2) = \begin{pmatrix} 0 & 1 \\ \beta_2 + 2\xi_1 + \xi_2 & \xi_1 \end{pmatrix}.$$

Substituting E_1 and computing the eigenvalues we yield:

$$\lambda_{1,2} = \frac{1}{4}\left(\sqrt{\beta_2^2 - 4\beta_1} + \beta_2 \pm \sqrt{(-\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2 - 16\sqrt{\beta_2^2 - 4\beta_1}}\right).$$

Since $\beta_2^2 > 4\beta_1$ we know $\sqrt{\beta_2^2 - 4\beta_1} > 0$. Also notice that if $\beta_2 < 0$ then $\sqrt{\beta_2^2 - 4\beta_1} + \beta_2 < 0$ and if $\beta_2 > 0$ then $\sqrt{\beta_2^2 - 4\beta_1} + \beta_2 > 0$. First suppose $(-\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2 > 16\sqrt{\beta_2^2 - 4\beta_1}$, meaning that our eigenvalues are real-valued. In this case we can see:

$$\begin{aligned}\sqrt{(-\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2 - 16\sqrt{\beta_2^2 - 4\beta_1}} &\leq \sqrt{(-\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2} \\ &= \left|-\sqrt{\beta_2^2 - 4\beta_1} - \beta_2\right| \\ &= \left|\sqrt{\beta_2^2 - 4\beta_1} + \beta_2\right|.\end{aligned}$$

This means that if $\beta_2 < 0$ then $\lambda_2 < \lambda_1 < 0$ which means E_1 will be a stable node. If $\beta_2 > 0$ then this gives us $0 < \lambda_2 < \lambda_1$ which indicates that E_1 will be a unstable node in this case. The origin $\beta = 0$ divides our curve T into two separate branches T_- and T_+ for $\beta_2 < 0$ and $\beta_2 > 0$ respectively. When we cross T_-, E_1 will be a stable node and when we cross T_+ E_1 will be an unstable node. This matches with what we expected to happen from the fold bifurcation. Now we look at what happens when $(-\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2 < 16\sqrt{\beta_2^2 - 4\beta_1}$. This means our eigenvalues will become imaginary indicating that E_1 undergoes a transition from a node to a focus. Looking at the real part $\sqrt{\beta_2^2 - 4\beta_1} + \beta_2$ we can see that for $\beta_2 < 0$ this focus will be stable and for $\beta_2 > 0$ the focus will be unstable.

If $\beta_1 = 0$ and $\beta_2 < 0$ we can see compute that:

$$\lambda_{1,2} = \pm\frac{1}{4}\sqrt{-16|\beta_2|} = \pm i\sqrt{|\beta_2|} = \pm i\sqrt{-\beta_2}.$$

Since both eigenvalues are purely imaginary (and each others complex conjugate) we can see that a Hopf bifurcation is occurring giving rise to a limit cycle. We actually computed the lyaponuv l_1 coefficient for several values of $\beta_2 < 0$ using the Matlab code Brusselator. For $\beta_2 = -0.01$ this gives $l_1 \approx -495.05$ and for $\beta_2 = 0.1$ we get $l_1 \approx -14.37$. For $\beta_2 = -0.5$ we yield $l_1 \approx -0.94$ and lastly for $\beta_2 = -1$ we get $l_1 = -\frac{1}{2}$. This gives us a pretty good indication (note this is not a rigorous proof) that we are dealing with a supercritical Hopf bifurcation, which means that this cycle will be stable.

Now we substitute E_2 into the Jacobi-matrix and calculate the corresponding eigenvalues:

$$\lambda_{1,2} = \frac{1}{4}\left(-\sqrt{\beta_2^2 - 4\beta_1} + \beta_2 \pm \sqrt{(\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2 + 16\sqrt{\beta_2^2 - 4\beta_1}}\right).$$

Again notice that since $\beta_2^2 > 4\beta_1$ we have $\sqrt{\beta_2^2 - 4\beta_1} > 0$. This means that if $\beta_2 < 0$ then $-\sqrt{\beta_2^2 - 4\beta_1} + \beta_2 < 0$ and if $\beta_2 > 0$ then $-\sqrt{\beta_2^2 - 4\beta_1} + \beta_2 > 0$. Now notice that we have:

$$\begin{aligned}\sqrt{(\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2 + 16\sqrt{\beta_2^2 - 4\beta_1}} &\geq \sqrt{(\sqrt{\beta_2^2 - 4\beta_1} - \beta_2)^2} \\ &= \left|\sqrt{\beta_2^2 - 4\beta_1} - \beta_2\right| \\ &= \left|\beta_2 - \sqrt{\beta_2^2 - 4\beta_1}\right|.\end{aligned}$$

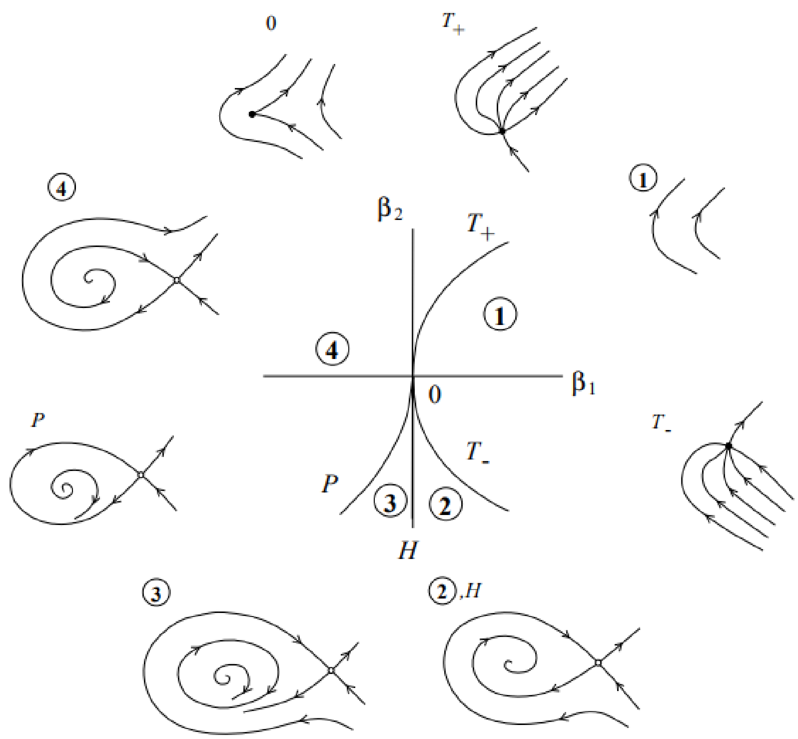

Figure 1: Bifurcation Diagram of system (1) (Source: [1])

This means that for any (β_1, β_2) with $\beta_2^2 > 4\beta_1$, we find $\lambda_2 < 0 < \lambda_1$. This means that whenever E_2 exists, it will be a saddle.

Now if we take a look at the bifurcation diagram we can go round in the bifurcation diagram near β_2. We will start in area **1** and go clockwise. We can see that if we cross T_- and go into area **2** then the system gains 2 equilibria, a saddle and stable. Then as we cross the Hopf bifurcation curve H into area **3**, we can see that our stable equilibrium (which has become a stable focus) bifurcates and a stable cycle comes into existence. Then as continue rotating clockwise the next thing we encounter is the fold bifurcation curve T^+, where our equilibria get destroyed again and we are back to the start. There is one problem with this though. We wonder what happened to our cycle. It must have gotten destroyed somewhere in region **3** because as we cross T^+ no cycles must remain. We only know two codimension 1 bifurcations that can destroy this cycle. This is the saddle homoclinic bifurcation and the saddle-node homoclinic bifurcation. Since the saddle-node equilibrium at our fold bifurcation cannot have a homoclinic orbit the only option is that a saddle homoclinic bifurcation occurs somewhere in region **3**. This gives rise to the saddle-homoclinic curve P dividing region **3** into two regions (which we will call **3** and **4**). With this knowledge we can now look at the following theorem:

Theorem 1. *There exists a unique smooth curve P in the (β_1, β_2)-plane which corresponds to the saddle-homoclinic bifurcation in system (1). It originates at $\beta = 0$ and has the representation:*

$$P = \{(\beta_1, \beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2), \beta_2 < 0\}.$$

Furthermore, for small $||\beta||$, system (1) has a unique cycle when β is in the region between the Hopf bifurcation curve H and the saddle-homoclinic curve P.

We will prove this theorem in the next section.

Unicity of the cycle

We divide the proof into 7 steps.

Step 1: Translating the system
For this entire proof we are only interested for values of β where $\beta_2^2 > 4\beta_1$ since our cycle only exists when the two equilibria E_1 and E_2 exist. We now translate the system such that our anti-saddle E_1 lies at the origin:

$$\begin{cases} \xi_1 = \eta_1 + \xi_1^0, \\ \xi_2 = \eta_2. \end{cases}$$

Here $\xi_1^0 = \frac{-\beta_2 - \sqrt{\beta_2^2 - 4\beta_1}}{2}$ like it was defined in the previous section. This gives $\dot{\xi_1} = \dot{\eta_1}$ and $\dot{\xi_2} = \dot{\eta_2}$. For the sake of notation we also introduce $\nu = \sqrt{\beta_2^2 - 4\beta_1}$ which is the distance between E_1 and E_2. Using this the second equation of system (1) becomes:

$$\begin{aligned}
\dot{\eta_2} &= \beta_1 + \beta_2\left(\eta_1 + \xi_1^0\right) + \left(\eta_1 + \xi_1^0\right)^2 - \eta_2\left(\eta_1 + \xi_1^0\right) \\
&= \beta_1 + \beta_2\left(\eta_1 + \frac{-\beta_2 - \nu}{2}\right) + \left(\eta_1 + \frac{-\beta_2 - \nu}{2}\right)^2 - \eta_2\left(\eta_1 + \xi_1^0\right) \\
&= \beta_1 + \beta_2\eta_1 - \frac{\beta_2^2}{2} - \frac{\beta_2\nu}{2} + \eta_1^2 - \eta_1\beta_2 - \eta_1\nu + \frac{\beta_2^2}{4} + \frac{\beta_2\nu}{2} + \frac{\nu^2}{4} - \eta_1\eta_2 - \eta_2\xi_1^0. \\
&= \frac{-\beta_2^2 + 4\beta_1}{4} + \frac{\nu^2}{4} + \eta_1(\eta_1 - \nu) - (\xi_1^0\eta_2 - \eta_1\eta_2) \\
&= \eta_1(\eta_1 - \nu) - (\xi_1^0\eta_2 - \eta_1\eta_2).
\end{aligned}$$

This means system (1) turns into the following:

$$\begin{cases} \dot{\eta_1} = \eta_2, \\ \dot{\eta_2} = \eta_1(\eta_1 - \nu) - (\xi_1^0\eta_2 + \eta_1\eta_2). \end{cases} \tag{3}$$

Step 2: Rescaling the system In this step we will rescale η_1, η_2 and t which will make the distance between the two equilibria equal to 1 and independent of any parameters. Define ζ_1, ζ_2 and τ as:

$$\zeta_1 = \frac{\eta_1}{\nu},\ \zeta_2 = \frac{\eta_2}{\nu^{3/2}},\ t = \frac{\tau}{\nu^{1/2}}. \tag{4}$$

This means:

$$\frac{d\zeta_1}{d\tau} = \nu^{-1}\frac{d\eta_1}{dt}\frac{dt}{d\tau} = \nu^{-3/2}\frac{d\eta_1}{d\tau}.$$

Similarly we calculate:

$$\frac{d\zeta_2}{d\tau} = \nu^{-3/2}\frac{d\eta_2}{dt}\frac{dt}{d\tau} = \nu^{-2}\frac{d\eta_2}{d\tau}.$$

From now on we we use the notation $\dot{\zeta_1}$, $\dot{\zeta_2}$ when taking the derivative with respect to our rescaled time τ. With this the first equation of system (3) becomes:

$$\dot{\zeta_1} = \zeta_2.$$

The second equation of system (3) will turn into:

$$\nu^2\dot{\zeta_2} = \nu\zeta_1(\nu\zeta_1 - \nu) - (\xi_1^0\nu^{3/2}\zeta_2 + \nu^{5/2}\zeta_1\zeta_2).$$

If we introduce new parameters:

$$\begin{cases} \gamma_1 = \xi_1^0 \nu^{-1/2}, \\ \gamma_2 = \nu^{1/2}, \end{cases}$$

and divide both sides by ν^2 we yield:

$$\dot{\zeta_2} = \zeta_1(\zeta_1 - 1) - (\gamma_1 \zeta_2 + \gamma_2 \zeta_1 \zeta_2).$$

Together this means we now have our rescaled system:

$$\begin{cases} \dot{\zeta_1} = \zeta_2, \\ \dot{\zeta_2} = \zeta_1(\zeta_1 - 1) - (\gamma_1 \zeta_2 + \gamma_2 \zeta_1 \zeta_2). \end{cases} \tag{5}$$

We can see if $\beta \to 0$ then clearly $\nu \to 0$ also looking at the formula of ξ_1^0 we can see also see that $\xi_1^0 \to 0$ as $\beta \to 0$. This means that $\gamma \to 0$. Note that since we did only rescale η_1, η_2 and time t linearly by positive constants we find that system (5) and system (3) must be orbitally equivalent. In the first step, the only thing we did is translate the entire system in such a way that our antisaddle now lies at the origin. This means system (3) is also equivalent to system (1). Putting these two together we get that system (5) is orbitally equivalent to system (1).

Step 3: The Hamiltonian system
Notice that if $\gamma = 0$ then system (5) becomes:

$$\begin{cases} \dot{\zeta_1} = \zeta_2, \\ \dot{\zeta_2} = \zeta_1(\zeta_1 - 1). \end{cases} \tag{6}$$

We can notice that this system is Hamiltonian with the Hamiltonian $H(\zeta_1, \zeta_2) = \frac{\zeta_1^2}{2} + \frac{\zeta_2^2}{2} - \frac{\zeta_1^3}{3}$. The idea is that if $||\gamma||$ is small we can view system (5) as a perturbed Hamiltonian system.

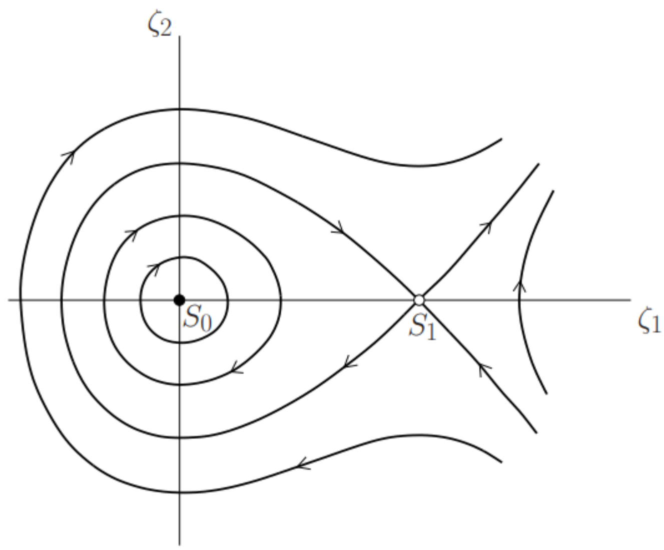

Figure 2: Phase portrait of the Hamiltonian system (6) (Source: [1])

In this step we will analyse the Hamiltonian properties of the system, which will helps us analyse the system for small γ. We can derive that $S_1 = (0, 0)$ and $S_2 = (1, 0)$ are the two equilibria of this system.

We can trace them all the way back to our equilibria in system (1), and find that S_1 corresponds to E_1 and S_2 to E_2. The Jacobi-matrix J' of this system is as follows:

$$J'(\zeta_1, \zeta_2) = \begin{pmatrix} 0 & 1 \\ 2\zeta_1 - 1 & 0 \end{pmatrix}.$$

We can substitute S_1 and S_2 into this expression and calculate the eigenvalues of the matrix. For S_1 this yields $\lambda_{1,2} = \pm i$ so S_1 is a centre, and for S_2 this yields $\lambda_{1,2} = \pm 1$ which implies that S_2 is a saddle. We can also compute that $H(S_1) = 0$ and $H(S_2) = \frac{1}{6}$. Since H is a Hamiltonian, we know that $\dot{H} = 0$ which means that the level sets of H consist of orbits of system (6). For the level set of the saddle $H = \frac{1}{6}$, we can find that $\zeta_2 = \pm\sqrt{\frac{1+2\zeta_1^3}{3} - \zeta_1^2}$, correspond to two separatrices of the saddle S_2. These two curves meet at the points $(1, 0)$ (our saddle S_2) and $(-\frac{1}{2}, 0)$ so these two curves together form a closed curve. Since there are no other equilibria in this system there can be no other equilibria on this closed curve we know that this closed curve corresponds to a homoclinic orbit. So now we have a homoclinic orbit with a centre inside. This means that the homoclinic orbit bounds the cycles inside. We can look at the level sets H corresponding to these cycles by studying the equation $H(\zeta_1, 0) = h$ for $\zeta_1 \in [0, 1]$. We can see that this gives $h \in [0, \frac{1}{6}]$. We can also see that $H(\zeta_1, 0)$ is monotone on this interval of ζ_1.

Step 4: The perturbed system.
Now let $||\gamma||$ be small instead of zero. This means that system (5) is no longer Hamiltonian. We do however still have the same two equilibria S_1 and S_2 as in the previous step. However since our Jacobi-matrix J' will now be different we can find that S_1 is no longer a centre. S_2 will stay as a saddle, but the homoclinic orbit from before can no longer be found for most values of γ, since for most values of γ the two separatrices no longer form a closed orbit like before. Now consider the part of the ζ_1-axis between S_1 and S_2 like in the previous step. Using the parametrisation $H(\zeta_1, 0) = h$ from the previous step, we take a point on this line with $h \in (0, \frac{1}{6})$ and look at the orbit that passes through this point. Since $||\gamma||$ is very small we assume that this orbit crosses the ζ_1-axis at least one more time in both forward and backward time. We call this intersection in backward time Z_- and in forward time Z_+. We can now look at the value H at the points Z_- and Z_+ (since the orbits do no longer have to be cycles, these values will often be different). We can now define the following function $\Delta(h, \gamma)$:

$$\Delta(h, \gamma) = H(Z_-) - H(Z_+). \tag{7}$$

We call this function the orbit split function. We extend this function to the end points by defining $\Delta(0, \gamma) = 0$ and for $h = \frac{1}{6}$ we define Z_- as the intersection with the ζ_1-axis of the stable separatrix of the saddle. We define Z_+ as the intersection of the unstable separatrix with the same axis. We can then apply the usual definition of Δ to $h = \frac{1}{6}$.

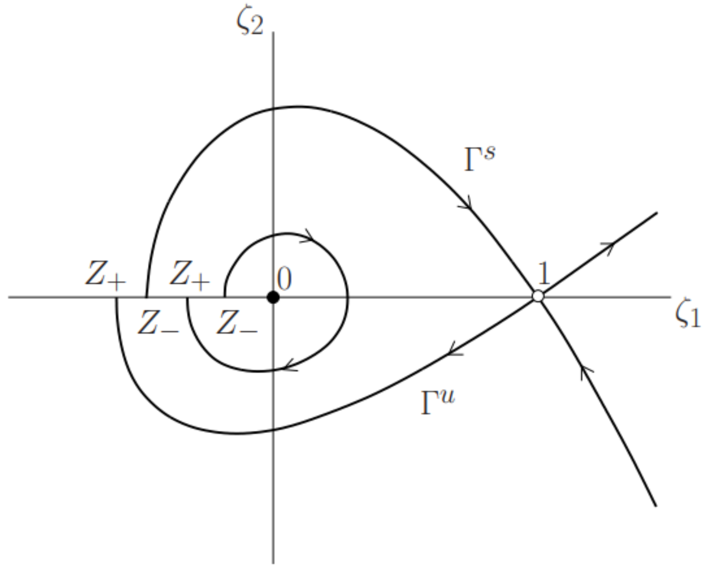

Figure 3: The perturbed Hamiltonian system and the split function (Source:[1])

Notice that this means that if for any γ we find that $\Delta(\frac{1}{6}, \gamma) = 0$ then we find a homoclinic orbit again. We call the curve of values of γ for which this happens $\mathcal{P}$ which is defined as follows:

$$\mathcal{P} = \{(\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_2 \geq 0,\ \Delta\left(\frac{1}{6}, \gamma\right) = 0\}. \tag{8}$$

We can also look at the equation $\Delta(h, \gamma) = 0$ for any $h \in (0, \frac{1}{6})$. Whenever (h, γ) satisfies this equation, we have found a cycle in system (5). This gives rise to the following curve in the (γ_1, γ_2)-plane:

$$\mathcal{L}_h = \{(\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_2 > 0, \Delta(h, \gamma) = 0\}. \tag{9}$$

Step 5: Approximating Δ. Our next step is to analyse our function Δ more closely and figure out a formula so we can study the curves we defined above. When $\gamma \neq 0$ we compute $\dot{H}$ (where we use that for $\gamma = 0$, $\dot{H} = 0$ so we are only interested in terms with a γ: in them):

$$\begin{aligned}
\dot{H} &= \frac{\partial H}{\partial \zeta_1}\dot{\zeta}_1 + \frac{\partial H}{\partial \zeta_2}\dot{\zeta}_2 \\
&= (\zeta_1 - \zeta_1^2)\dot{\zeta}_1 + \zeta_2\dot{\zeta}_2 \\
&= -\zeta_2(\gamma_1\zeta_2 + \gamma_2\zeta_1\zeta_2) \\
&= -\gamma_1\zeta_2^2 - \gamma_2\zeta_1\zeta_2^2.
\end{aligned}$$

Define Γ as part of an orbit from system (5) ranging from Z_- to Z_+. Using the fundamental theorem of calculus we find:

$$\begin{aligned}\Delta(h,\gamma) &= \int_{\tau_{Z_+}}^{\tau_{Z_-}} \dot{H}\ d\tau = -\int_{\tau_{Z_-}}^{\tau_{Z_+}} \dot{H}\ d\tau \\ &= \int_{\tau_{Z_-}}^{\tau_{Z_+}} (\gamma_1\zeta_2^2 + \gamma_2\zeta_1\zeta_2^2)\ d\tau \\ &= \int_\Gamma (\gamma_1\zeta_2\dot{\zeta_1} + \gamma_2\zeta_1\zeta_2\dot{\zeta_1})\ d\tau \\ &= \int_\Gamma (\gamma_1\zeta_2\dot{\zeta_1} + \gamma_2\zeta_1\zeta_2\dot{\zeta_1}\ )d\tau \\ &= \gamma_1 \int_\Gamma \zeta_2\ d\zeta_1 + \gamma_2 \int_\Gamma \zeta_1\zeta_2\ d\zeta_1.\end{aligned}$$

Here the orientation of Γ is given by the direction as we increase time. For $h = \frac{1}{6}$ is worth to note that by integrating over Γ we mean integrating over the unstable and stable separatrices separately and summing them. This expression is exact, but comes with the problem that we have no explicit formula for our orbits Γ. However, when $||\gamma||$ is small, the orbits will only differ ever so slightly from the closed orbits from system (6). These orbits are just level sets from the Hamiltonian $H(\zeta) = h$. So if we use this approximation we can write:

$$\Delta(h,\gamma) = \gamma_1 \int_{H(\zeta)=h} \zeta_2\ d\zeta_1 + \gamma_2 \int_{H(\zeta)=h} \zeta_1\zeta_2\ d\zeta_1 + o(||\gamma||).$$

We will now name:

$$I_1(h) = \int_{H(\zeta)=h} \zeta_2\ d\zeta_1, \tag{10}$$

and:

$$I_2(h) = \int_{H(\zeta)=h} \zeta_1\zeta_2\ d\zeta_1. \tag{11}$$

Step 6: Uniqueness of the limit cycle In this step we will prove that for certain values of γ there will be a unique limit cycle in system (5). We first use the implicit function theorem. For $\mathcal{L}_h$, corresponding to $h \in (0, \frac{1}{6})$, we have the equation $\Delta(h,\gamma) = 0$. We have the same equation for $\mathcal{P}$, corresponding to $h = \frac{1}{6}$. Since $D\gamma_1(h,\gamma) = I_1(h) \neq 0$ for any point on $\mathcal{P}$ or $\mathcal{L}_h$. We find that the implicit function theorem tells us that these two curves exist. It also tells us that $\mathcal{L}_h$ for $h \in (0, \frac{1}{6})$ and $\mathcal{P}$ for $h = \frac{1}{6}$ can be represented by:

$$\gamma_1(h,\gamma_2) = -\frac{I_2(h)}{I_1(h)}\gamma_2 + o(|\gamma_2|),\ \ \gamma_2 \geq 0. \tag{12}$$

We can find a more explicit from of (10) for $\mathcal{P}$. To do this we first prove the following lemma:

Lemma 2. $Q(\frac{1}{6}) = \frac{1}{7}$.

Proof. To compute this recall that we found that our homoclinic orbit could be parametrised by:

$$\zeta_2 = \pm\sqrt{\frac{1+2\zeta_1^3}{3} - \zeta_1^2}.$$

Also recall that these two parts meet at the points $(-\frac{1}{2}, 0)$ and $(1, 0)$. We can now use the symmetry of the homoclinic orbit to compute that:

$$I_1\left(\frac{1}{6}\right) = \int_{H(\zeta)=\frac{1}{6}} \zeta_2\ d\zeta_1 = \int_{-\frac{1}{2}}^{1} \sqrt{\frac{1+2\zeta_1^3}{3} - \zeta_1^2}\ d\zeta_1 = \frac{6}{5}.$$

Similarly we compute:

$$I_2\left(\frac{1}{6}\right) = \int_{H(\zeta)=\frac{1}{6}} \zeta_1\zeta_2 \; d\zeta_1 = \int_{-\frac{1}{2}}^{1} \zeta_1 \sqrt{\frac{1+2\zeta_1^3}{3} - \zeta_1^2} \; d\zeta_1 = \frac{6}{35}.$$

This gives $Q(\frac{1}{6}) = \frac{6}{35}\frac{5}{6} = \frac{1}{7}$, proving the claim. $\square$

With this we can conclude that $\mathcal{P}$ has the following characterisation:

$$\mathcal{P} = \{(\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_1 = -\frac{1}{7}\gamma_2 + o(|\gamma_2|), \gamma_2 \geq 0\}. \tag{13}$$

Now if we start at $h = 0$ and increase h to $\frac{1}{6}$. The curve $\mathcal{L}_h$ moves from the vertical half axis $\{\gamma \in \mathbb{R}^2 : \gamma_1 = 0, \gamma_2 \geq 0\}$ towards curve $\mathcal{P}$. If we can prove that this motion is monotonous as we increase h we know that there must be a cycle in this enclosed area and it must be unique. This monotonicity of this motion is captured in the following function:

$$Q(h) = \frac{I_2(h)}{I_1(h)},$$

for $h \in (0, \frac{1}{6}]$. Note that this function is smooth for these values since we are just integrating two different polynomials over smooth curves and then dividing these by each other. Note that $\lim_{h\downarrow 0} \frac{I_2(h)}{I_1(h)} = 0$. So we extend this function in a smooth way to 0 by defining $Q(0) = 0$. In the book *Elements of Applied Bifurcation Theory* [1], they made the following numerical plot of Q:

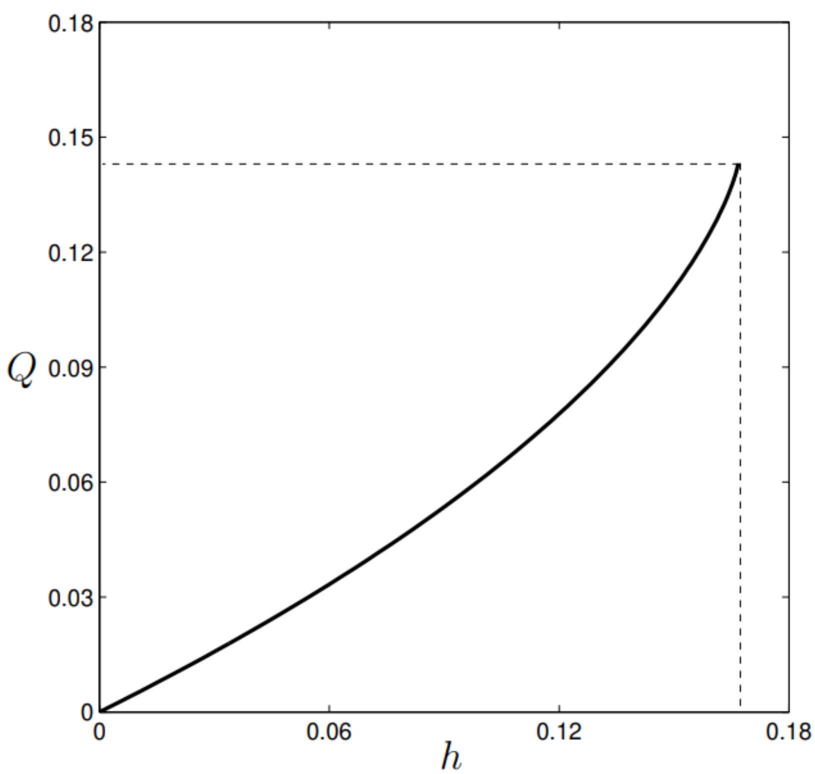

Figure 4: Numerical plot of Q (Source: [1])

In here we can clearly see that Q is monotonous as we increase h from 0 to $\frac{1}{6}$.

Before we move on to proving analytically that Q is monotonous, we want to make sure that there are no other cycles somewhere in system (5). For any values of γ outside this enclosed area between $\mathcal{P}$ and the vertical half-axis. there cannot exist a cycle. If there were to be a closed orbit outside for any γ outside this area then this orbit has to cross the ζ_1 axis between S_1 and S_2. This is true since we know from index theory that any closed orbit of a dynamical system, has to enclose at least one fixed point. We cannot have any closed orbits around just the saddle, since a saddle has index -1 and any closed orbit of a dynamical system must have index 1. We also cannot have any closed orbits around both our equilibria since then according to index theory the index of this closed orbit should be equal to the sum of the indices of the two equilibria. Since a focus has index 1 and the saddle has index -1 these will sum to be 0. Since this is not equal to 1 we conclude that this is also not an option. By checking the zeroes of our function Δ we ensure that there are no cycles that cross the axis between S_1 and S_2 other than the ones we have already found.

We now move on to proving that Q is monotonous, which is what the following lemma will state:

Lemma 3. *For $h \in [0, \frac{1}{6}]$ we have $\frac{dQ}{dh} > 0$.*

If we can prove that this lemma is true then we have proved the unicity of the cycle in area enclosed by the vertical half-axis and $\mathcal{P}$.

Proof of Lemma 2:
To prove this lemma we make use of a few propositions:

Proposition 4. *The integrals $I_1(h)$ and $I_2(h)$ satisfy the following system of differential equations:*

$$\begin{cases} h(h-\frac{1}{6})\dot{I}_1 = (\frac{5}{6}h - \frac{1}{6})I_1 + \frac{7}{36}I_2, \\ h(h-\frac{1}{6})\dot{I}_2 = -\frac{1}{6}hI_1 + \frac{7}{6}hI_2, \end{cases} \tag{14}$$

where the dots represent taking the derivative with respect to h.

Proof. Let $h \in (0, \frac{1}{6})$. We can now take the equation $H(\zeta) = h$ which corresponds to a closed orbit of system (6):

$$\frac{\zeta_1^2}{2} + \frac{\zeta_2^2}{2} - \frac{\zeta_1^3}{3} = h. \tag{15}$$

We will now consider ζ_2 as a function of ζ_1 and h. If we differentiate equation (15) on both sides with respect to h, we yield:

$$\zeta_2 \frac{\partial \zeta_2}{\partial h} = 1.$$

We rewrite this to $\frac{\partial \zeta_2}{\partial \zeta} = \frac{1}{\zeta_2}$. This means if take the derivative of I_1 and I_2 with respect to h and take this derivative inside the integral (which we can do since the integrands are smooth), we can use this identity:

$$\frac{dI_1}{dh} = \int_{H(\zeta)=h} \frac{\partial}{\partial h}\zeta_2 \; d\zeta_1 = \frac{d\zeta_1}{\zeta_2}, \tag{16}$$

and:

$$\frac{dI_2}{dh} = \int_{H(\zeta)=h} \frac{\partial}{\partial h}\zeta_1\zeta_2 \; d\zeta_1 = \int_{H(\zeta)=h} \frac{\zeta_1 d\zeta_1}{\zeta_2}. \tag{17}$$

We can also differentiate (15) with respect to ζ_1. This gives:

$$\zeta_1 + \zeta_2 \frac{\partial \zeta_2}{\partial \zeta_1} - \zeta_1^2 = 0.$$

We rewrite this to $\zeta_1^2 = \zeta_1 + \zeta_2 \frac{\partial \zeta_2}{\partial \zeta_1}$. We now multiply this equation by $\zeta_1^m \zeta_2^{-1}$, where $m = 0, 1$ or 2, and we integrate over ζ_1:

$$\int_{H(\zeta)=h} \frac{\zeta_1^{m+2} d\zeta_1}{\zeta_2} = \int_{H(\zeta)=h} \frac{\zeta_1^{m+1} d\zeta_1}{\zeta_2} + \int_{H(\zeta)=h} \zeta_1^m \frac{\partial \zeta_2}{\partial \zeta_1}\ d\zeta_1. \tag{18}$$

We simplify this third integral using partial integration:

$$\int_{H(\zeta)=h} \zeta_1^m \frac{\partial \zeta_2}{\partial \zeta_1}\ d\zeta_1 = \zeta_1^m \zeta_2 \Big|_{H(\zeta)=h} - \int_{H(\zeta)=h} m \zeta_1^{m-1} \zeta_2\ d\zeta_1.$$

Note that:

$$\zeta_1^m \zeta_2 \Big|_{H(\zeta)=h} = 0,$$

because of the symmetry of the curve $H(\zeta) = h$ in the ζ_1-axis. Using this we can simplify (18) to the following:

$$\int_{H(\zeta)=h} \frac{\zeta_1^{m+2} d\zeta_1}{\zeta_2} = \int_{H(\zeta)=h} \frac{\zeta_1^{m+1} d\zeta_1}{\zeta_2} - m \int_{H(\zeta)=h} \zeta_1^{m-1} \zeta_2\ d\zeta_1. \tag{19}$$

Using (15) and (16) we find:

$$\begin{aligned} h \frac{dI_1}{dh} &= h \frac{d\zeta_1}{\zeta_2} \\ &= \frac{1}{2} \int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} + \frac{1}{2} \int_{H(\zeta)=h} \zeta_2\ d\zeta_1 - \frac{1}{3} \int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} = (*). \end{aligned}$$

Now use the definition of I_1 (10) and equation (19) for $m = 1$:

$$\begin{aligned} (*) &= \frac{1}{2} I_1 + \frac{1}{2} \int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} - \frac{1}{3} \int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} \\ &= \frac{1}{2} I_1 + \frac{1}{2} \int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} - \frac{1}{3} \int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} + \frac{1}{3} \int_{H(\zeta)=h} \zeta_2\ d\zeta_1 \\ &= \frac{5}{6} I_1 + \frac{1}{6} \int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} = (**). \end{aligned}$$

We can now apply equation (19) once more for $m = 0$ and use (17):

$$(**) = \frac{5}{6} I_1 + \frac{1}{6} \int_{H(\zeta)=h} \frac{\zeta_1 d\zeta_1}{\zeta_2} = \frac{5}{6} I_1 + \frac{1}{6} \frac{dI_2}{dh}.$$

So we have found:

$$h \frac{dI_1}{dh} = \frac{5}{6} I_1 + \frac{1}{6} \frac{dI_2}{dh}. \tag{20}$$

We can now follow a similar thought process for $h \frac{dI_2}{dh}$. Using (15) and (17) we find:

$$\begin{aligned} h \frac{dI_2}{dh} &= \int_{H(\zeta)=h} \frac{\zeta_1 d\zeta_1}{\zeta_2} \\ &= \frac{1}{2} \int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} + \frac{1}{2} \int_{H(\zeta)=h} \zeta_1 \zeta_2\ d\zeta_1 - \frac{1}{3} \int_{H(\zeta)=h} \frac{\zeta_1^4 d\zeta_1}{\zeta_2} = (\Box). \end{aligned}$$

Now using equation (19) for $m = 2$ and for $m = 1$ together with (11) we yield:

$$\begin{aligned}
(\Box) &= \frac{1}{2}I_2 + \frac{1}{2}\int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} - \frac{1}{3}\int_{H(\zeta)=h} \frac{\zeta_1^4 d\zeta_1}{\zeta_2} \\
&= \frac{1}{2}I_2 + \frac{1}{2}\int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} - \frac{1}{3}\int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} + \frac{2}{3}\int_{H(\zeta)=h} \zeta_1\zeta_2 \; d\zeta_1 \\
&= \frac{7}{6}I_2 + \frac{1}{6}\int_{H(\zeta)=h} \frac{\zeta_1^3 d\zeta_1}{\zeta_2} \\
&= \frac{7}{6}I_2 + \frac{1}{6}\int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} - \frac{1}{6}\zeta_2 \; d\zeta_1 = (\Box\Box).
\end{aligned}$$

Finally, using (10) and (17) together with (19) for $m = 0$ together with :

$$\begin{aligned}
(\Box\Box) &= \frac{7}{6}I_2 - \frac{1}{6}I_1 + \frac{1}{6}\int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} \\
&= \frac{7}{6}I_2 - \frac{1}{6}I_1 + \frac{1}{6}\int_{H(\zeta)=h} \frac{\zeta_1^2 d\zeta_1}{\zeta_2} \\
&= \frac{7}{6}I_2 - \frac{1}{6}I_1 + \frac{1}{6}\frac{dI_2}{dh}.
\end{aligned}$$

So from this we yield:

$$h\frac{dI_2}{dh} = \frac{7}{6}I_2 - \frac{1}{6}I_1 + \frac{1}{6}\frac{dI_2}{dh}.$$

We can rewrite this to:

$$(h - \frac{1}{6})\dot{I}_2 = -\frac{1}{6}I_1 + \frac{7}{6}I_2. \tag{21}$$

Now we multiply equation (20) by $(h - \frac{1}{6})$ on both sides and substitute (21) in there:

$$h(h - \frac{1}{6})\dot{I}_1 = (h - \frac{1}{6})\frac{5}{6}I_1 + \frac{1}{6}(-\frac{1}{6}I_1 + \frac{7}{6}I_2) = (\frac{5}{6}h - \frac{1}{6})I_1 + \frac{7}{36}I_2.$$

This is the first equation of the system (14). We obtain the second equation by multiplying (21) by h on both sides. With this the proposition is proved. □

We now move on to the next proposition we need:

Proposition 5. *The function $Q(h) = \frac{I_1(h)}{I_2(h)}$ satisfies the Riccati equation:*

$$h\left(h - \frac{1}{6}\right)\dot{Q} = -\frac{7}{36}Q^2 + \left(\frac{h}{3} + \frac{1}{6}\right)Q - \frac{h}{6}. \tag{22}$$

Proof. We start by using system (14) we just proved together with the quotient rule and the definition:

$$\begin{aligned}
h\left(h-\frac{1}{6}\right)\dot{Q} &= h\left(h-\frac{1}{6}\right)\left(\frac{I_1\dot{I}_2 - I_2\dot{I}_1}{I_1^2}\right)\\
&= h\left(h-\frac{1}{6}\right)\left(\frac{\dot{I}_2}{I_1} - \frac{I_2\dot{I}_1}{I_1^2}\right)\\
&= -\frac{h}{6}\frac{I_1}{I_1} + \frac{7h}{6}\frac{I_2}{I_1} - \frac{I_2}{I_1}\left(\left(\frac{5h}{6}-\frac{1}{6}\right)\frac{I_1}{I_1} + \frac{7}{36}\frac{I_2}{I_1}\right)\\
&= -\frac{h}{6} + \frac{7h}{6}Q - Q\left(\frac{5h}{6} - \frac{1}{6} + \frac{7}{36}Q\right)\\
&= -\frac{7}{36}Q^2 + \left(\frac{h}{3}+\frac{1}{6}\right)Q - \frac{h}{6}.
\end{aligned}$$

This proves the proposition. □

If we take the Taylor series of Q at $h = 0$ we get $Q(h) = Q(0) + \dot{Q}(0)h + \frac{\ddot{Q}(0)}{2}h^2 + \cdots = \alpha h + O(h^2)$. If we substitute this into the Ricatti equation (22), then we yield:

$$h\left(h-\frac{1}{6}\right)\alpha = -\frac{7}{36}\alpha^2 h^2 + \left(\frac{h}{3}+\frac{1}{6}\right)\alpha h - \frac{h}{6} + O(h^2),$$

which simplifies to:

$$\frac{1}{3}\alpha h - \frac{h}{6} + O(h^2) = 0.$$

Solving $\frac{1}{3}\alpha h - \frac{h}{6} = 0$ for any value of $h \in [0, \frac{1}{6}]$ implies that $\alpha = \frac{1}{2}$. This means that $\dot{Q}(0) = \alpha = \frac{1}{2} > 0$. This property of Q will come in handy when proving the following proposition:

Proposition 6. *For all $h \in (0, \frac{1}{6})$, the function Q satisfies $0 \leq Q(h) \leq \frac{1}{7}$.*

Proof. We already know $Q(0) = 0$ and $Q(\frac{1}{6}) = \frac{1}{7}$. This means we still need to check what happens for $h \in (0, \frac{1}{6})$. Since $\dot{Q}(0) > 0$ and $Q(0) = 0$ we know that $Q(h)$ is positive for small $h > 0$. Suppose $\bar{h} \in (0, \frac{1}{6}$ is h-coordinate of the first intersection of the graph of $Q(h)$ with the positive h-axis. This means that $Q(\bar{h}) = 0$ and $Q(h) > 0$ for every $h \in (0, \bar{h})$. These two together imply that $\dot{Q}(\bar{h}) < 0$. If we now substitute $\bar{h}$ into the Ricatti equation (22) and notice that $-\frac{\bar{h}}{6} < 0$ we get:

$$\bar{h}\left(\bar{h}-\frac{1}{6}\right)\dot{Q}(\bar{h}) = -\frac{7}{36}Q(\bar{h})^2 + \left(\frac{h}{3}+\frac{1}{6}\right)Q(\bar{h}) - \frac{\bar{h}}{6} = -\frac{\bar{h}}{6} < 0.$$

Notice that $\bar{h} > 0$ and $(\bar{h} - \frac{1}{6}) < 0$. This means the inequality above can only hold if $\dot{Q}(\bar{h}) > 0$ which is a contradiction. This means that $Q(h) \geq 0$. Now suppose $\bar{h} \in (0, \frac{1}{6})$ is the first intersection of the graph of $Q(h)$ with the line $Q = \frac{1}{7}$. This means that $Q(\bar{h}) = \frac{1}{7}$ and $0 \leq Q(h) < \frac{1}{7}$ for all $h \in (0, \bar{h})$. Together these two imply that $\dot{Q}(\bar{h}) > 0$. Now we substitute $\bar{h}$ into the Ricatti equation (22):

$$\begin{aligned}
\bar{h}\left(\bar{h}-\frac{1}{6}\right)\dot{Q}(\bar{h}) &= -\frac{7}{36}Q(\bar{h})^2 + \left(\frac{h}{3}+\frac{1}{6}\right)Q(\bar{h}) - \frac{\bar{h}}{6}\\
&= \frac{7}{36}\frac{1}{7^2} + \left(\frac{h}{3}+\frac{1}{6}\right)\frac{1}{7} - \frac{\bar{h}}{6} = \frac{5}{42}\left(\frac{1}{6}-\bar{h}\right).
\end{aligned}$$

Notice that $\bar{h} > 0$, $\bar{h} - \frac{1}{6} < 0$ and $\frac{1}{6} - \bar{h} > 0$. From this we yield:

$$\bar{h}\left(\bar{h}-\frac{1}{6}\right)\dot{Q}(\bar{h}) = \frac{5}{42}\left(\frac{1}{6}-\bar{h}\right) < 0,$$

which can only hold if $\dot{Q}(\bar{h}) < 0$ giving a contradiction. This proves the proposition. □

Now lastly, we need to figure out if there are any points $0 < h < \frac{1}{6}$ where $\dot{Q}$ vanishes. Suppose $\bar{h} \in (0, \frac{1}{6})$ is such a point. This means $\dot{Q}(\bar{h}) = 0$. We now compute the second derivative at this point. Taking the derivative with respect to h on both sides of the Ricatti equation (22) we get:

$$(h - \frac{1}{6})\dot{Q} + h\dot{Q} + h(h - \frac{1}{6})\ddot{Q}(h) = -\frac{7}{18}Q\dot{Q} + \frac{1}{3}Q + \left(\frac{h}{3} + \frac{1}{6}\right)\dot{Q} - \frac{1}{6}.$$

Rewriting this equation and applying that $\dot{Q}(\bar{h}) = 0$ gives:

$$\ddot{Q}(\bar{h})\Big|_{\dot{Q}(\bar{h})=0} = \frac{1}{3}\left(Q - \frac{1}{2}\right).$$

The proposition 6 gives $Q - \frac{1}{2} < 0$ so:

$$\ddot{Q}(\bar{h})\Big|_{\dot{Q}(\bar{h})=0} = \frac{1}{3}\left(Q - \frac{1}{2}\right) < 0.$$

This means that at any point $\bar{h}$ where $\dot{Q}(\bar{h}) = 0$ that $\ddot{Q}(\bar{h}) < 0$. This means any extrema is a minimum. We know $Q(0) = 0$ and $\dot{Q}(0) > 0$. This means that if Q were to have a minimum $\bar{h}$ between $h = 0$ and $h = \frac{1}{6}$ then it must first have a maximum, which is not possible. This means no such point $\bar{h}$ can exist. Since $Q(\frac{1}{6}) = \frac{1}{7} = \max_{0 \leq h \leq \frac{1}{6}} Q(h)$, which means that $\dot{Q}(\frac{1}{6}) > 0$. We now know $\dot{Q}(0) > 0$, $\dot{Q}(h) \neq 0$ for all $h \in (0, \frac{1}{6}]$. Since Q is smooth, this means that $\dot{Q}$ cannot be negative for any h, which implies that $\dot{Q} > 0$ for all $h \in [0, \frac{1}{6})$. This proves lemma 3. □

Now that we have proven this lemma, we know that the cycle in system (5) is unique.

Step 7: Returning to the original parameters The only the thing that remains is to translate the results we have found to our original parameters in system (1). To do this we have to study the map $\gamma(\beta)$. Recall that:

$$\begin{cases} \gamma_1 = \xi_1^0 \nu^{-1/2}, \\ \gamma_2 = \nu^{1/2}. \end{cases} \tag{23}$$

Note that for $\beta_2^2 > 4\beta_1$, ξ_1^0 and ν are smooth functions of γ. Since for these values of β $\nu(\beta) > 0$ we also get that $\nu^{1/2}$ and $\nu^{-1/2}$ will also depend smoothly on β. This means that $\gamma(\beta)$ is a smooth function. The fact that $\nu > 0$ also gives $\gamma_2 = \nu^{1/2} > 0$ and this means that γ maps the region:

$$\{(\beta_1, \beta_2) \in \mathbb{R}^2 : \beta_2^2 > 4\beta_1\},$$

homeomorphically on the upper half-plane:

$$\{(\gamma_1, \gamma_2) \in \mathbb{R}^2 : \gamma_2 \geq 0\}.$$

This means that γ is a smooth diffeomorphism. This implies that $D\gamma(\beta)$ is locally invertible (opmerking 2.7 [2]). Since we are only interested in values of β near the origin, this is sufficient to apply the inverse function theorem. We can rewrite (23) to:

$$-\frac{\beta_2}{2} - \frac{\gamma_2^2}{2} = \gamma_1\gamma_2,$$
$$\beta_2^2 - 4\beta_1 = \gamma_2^4.$$

Using the inverse function theorem we yield that these equations define a smooth function $\beta(\gamma)$. First we rewrite the first equation to:

$$\beta_2 = -\gamma_2^2 - 2\gamma_1\gamma_2 = -\gamma_2(2\gamma_1 + \gamma_2).$$

We can rewrite the second equation to:

$$\beta_1 = \frac{1}{4}(\beta_2^2 - \gamma_2^4).$$

Substituting the expression we got for β_2 in here we yield:

$$\beta_1 = \frac{1}{4}([-\gamma_2(2\gamma_1 + \gamma_2)]^2 - \gamma_2^4) = \gamma_1\gamma_2^2(\gamma_1 + \gamma_2) + o(||\gamma||^4)$$

So these two together form:

$$\begin{cases} \beta_1 = \gamma_1\gamma_2^2(\gamma_1 + \gamma_2) + o(||\gamma||^4), \\ \beta_2 = -\gamma_2(2\gamma_1 + \gamma_2). \end{cases} \tag{24}$$

Using this we can now start to derive:

$$P = \{(\beta_1, \beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2), \beta_2 < 0\}. \tag{25}$$

So from (13) we know for $\gamma_2 \geq 0$:

$$\gamma_1 = -\frac{1}{7}\gamma_2 + o(|\gamma_2|).$$

Which we rewrite to:

$$\gamma_2 + 7\gamma_1 + o(|\gamma_2|) = 0.$$

Now we multiply both sides by $(6\gamma_2 + 7\gamma_1)\gamma_2^2$:

$$(\gamma_2 + 7\gamma_1)(6\gamma_2 + 7\gamma_1)\gamma_2^2 + o(||\gamma||^4) = 6\gamma_2^4 + 49\gamma_1\gamma_2^3 + 49\gamma_1^2\gamma_2^2 + o(||\gamma||^4) = 0.$$

We rewrite this to:

$$25\gamma_1^2\gamma_2^2 + 25\gamma_1\gamma_2^3 = -24\gamma_1^2 + \gamma_2^2 - 24\gamma_1\gamma_2^3 - 6\gamma_2^4 + o(||\gamma||^4).$$

We need to rewrite this one step further:

$$25\gamma_1\gamma_2^2(\gamma_1 + \gamma_2) = -6(-\gamma_2(\gamma_1 + \gamma_2))^2 + o(||\gamma||^4).$$

Now we can use (24) to conclude:

$$\beta_1 = -\frac{6}{25}\beta_2^2 + (o(\beta_2^2)),$$

where we used that $o(\beta_2^2) = o(||\gamma||^4)$. This proves that (25) is the right representation of P. Note that (24) maps the the vertical half-axis in the
$(\gamma_1, \gamma_2)-$plane to the Hopf bifurcation line H in the $(\beta_1, \beta_2)-$plane. Since system (1) and (5) are orbitally equivalent, we now know that there exist a unique cycle in system (1) in the region enclosed by H and P in the $(\beta_1, \beta_2)-$plane. This concludes the proof of the theorem.

References

[1] Yuri A. Kuznetsov, *Elements of Applied bifurcation Bifurcation Theory*, Springer-Verlag, New York, Second edition, 1998.

[2] F. Ziltener, *Analyse in meer variabelen, WISB212*, April 20, 2023.