INLDS Practicum 3

For each planar system below, construct its phase portrait numerically (using the `MATLAB` tool `pplane9`) and then try to prove its essential features analytically.

Exercises

Ex.1 Lotka-Volterra system

$$\begin{cases} \dot{x} &=& x - xy, \\ \dot{y} &=& -y + xy, \end{cases} \tag{1}$$

where $x, y \geq 0$.

Hint: Introduce new variables $q = \ln x$ and $p = \ln y$ and prove that the resulting (q, p)-system is Hamiltonian.

Ex.2 Van der Pol oscillator as a perturbed Hamiltonian system

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& -x + \varepsilon y(1 - x^2), \end{cases} \tag{2}$$

with $\varepsilon = 0, 0.01, 0.1$ and 1.0. Compute analytically the asymptotic radius of the cycle as $\varepsilon \to 0$.

Ex.3 Homoclinic orbit in blow-up of Bogdanov-Takens normal form

Consider

$$\begin{cases} \dot{x} &=& y, \\ \dot{y} &=& \alpha + \beta y + x^2 + xy. \end{cases} \tag{3}$$

After the (singular) rescaling $x = \varepsilon^2 u$, $y = \varepsilon^3 v$, $\alpha = -4\varepsilon^4$, $\beta = \varepsilon^2 \tau$, $\varepsilon t = s$, we obtain

$$\begin{cases} \dot{u} &=& v, \\ \dot{v} &=& -4 + u^2 + \varepsilon v(\tau + u). \end{cases} \tag{4}$$

(a) Show that for $\epsilon = 0$ the new system (4) is Hamiltonian. Use the Hamiltonian to show that there is a homoclinic orbit to a saddle.

(b) For small $\varepsilon \geq 0$ there is a curve $\tau = \tau_{\text{HOM}}(\varepsilon)$ such that the perturbed system (4) has a homoclinic orbit to the saddle. Approximate $\tau_0 := \tau_{\text{HOM}}(0)$ numerically, e.g. find the homoclinic parameter value τ_{HOM} at several constant values of $\varepsilon > 0$ by simulating (4) and observing convergence.

(c) **Challenge**: You may use $(u_0(t), v_0(t)) = 2(1 - 3\text{sech}^2(t), 6\text{sech}^2(t)\tanh(t))$ on the homoclinic orbit at $\varepsilon = 0$ to find τ_0 analytically with Theorem 2.22 in *Applied Nonlinear Dynamics*. Use the obtained τ_0 to derive a quadratic approximation of the saddle homoclinic bifurcation curve $\alpha = \alpha_{\text{HOM}}(\beta)$ in (3).

Homework

Hand-in exercise is number 3.