# 1 Poincaré–Bendixson Theorem

1. Give the definition of the $\omega$–limit sets for bounded orbits of smooth $n$–dimensional ODEs and establish their basic properties.

2. Characterize the $\omega$–limit sets of bounded orbits of smooth planar ODEs, i.e. prove the Poincaré–Bendixson theorem in $\mathbb{R}^2$.

3. Give examples that were not used in the lecture notes or during the practicum.

*References*:


[1] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Chapter 4]
`http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect6_7.pdf`

[2] F. Verhulst, *Nonlinear Differntial Equations and Dynamical Sysyems*, 2nd ed. Springer (1996) [Section 4.2]

[3] J.D. Meiss, *Differential Dynamical Systems.* SIAM (2007) [Sections 4.9, 6.6, and 6.7]

## 2 Uniqueness of the limit cycle near Hopf bifurcation

Consider the following smooth orbital normal form for the supercritical Hopf bifurcation:

$$\dot{w} \;=\; (\alpha + \mathrm{i})w \;-\; w|w|^2 \;+\; O(|w|^4)\ , \quad w \in \mathbb{C}, \tag{2.1}$$

where the $O(|w|^4)$–terms can smoothly depend on $\alpha \in \mathbb{R}$.

1. Derive the cubic Taylor expansion of the parameter-dependent Poincaré mapping of (2.1) defined on the half-axis $\mathrm{Re}\, w \geq 0$ near $w = 0$. Show that it is independent of the $O(|w|^4)$–terms in (2.1).

2. Prove that the Poincaré mapping has a unique stable positive fixed point when $\alpha > 0$. Conclude from this that a unique stable limit cycle bifurcates from the origin in (2.1) with any $O(|w|^4)$–terms.

3. Give an alternative proof of the uniqueness of the cycle using the Poincaré–Bendixson–Dulac theory, i.e. by constructing a trapping annulus for (2.1), where the corresponding vector field has negative divergence.

*References*:


[1] Yu.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 4rd ed., Springer (2023) [Section 3.7]

[2] Yu.A. Kuznetsov, *Applied Nonlinear Dynamics.* Utrecht University & University of Twente (2023) [Section 3.3, Exercise 3]

# 3 Uniqueness of the limit cycle near BT bifurcation

Consider the Bogdanov normal form for the Bogdanov–Takens bifurcation

$$\begin{cases} \dot{\xi}_1 &= \xi_2, \\ \dot{\xi}_2 &= \beta_1 + \beta_2\xi_1 + \xi_1^2 - \xi_1\xi_2. \end{cases} \tag{3.1}$$

1. Show that for $\beta_2^2 > 4\beta_1$ system (3.1) is orbitally equivalent to a perturbed Hamiltonian system

   $$\begin{cases} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_1(\zeta_1 - 1) \ - \ (\gamma_1\zeta_2 + \gamma_2\zeta_1\zeta_2) \end{cases} , \tag{3.2}$$

   where $\gamma_j = \gamma_j(\beta) \to 0$ as $\beta \to 0$.

2. Study periodic and saddle homoclinic orbits in (3.2). In particular, prove that it has exactly one cycle for small $\|\gamma\|$ between the vertical half-axis

   $$H \ = \ \{\gamma : \gamma_1 = 0, \gamma_2 \geq 0\}$$

   and a curve

   $$P \ = \ \{\gamma : \gamma_1 = -\frac{1}{7}\gamma_2 + o(|\gamma_2|), \gamma_2 \geq 0\} \ .$$

3. Discuss the complete bifurcation diagram of the original system (3.1) in the $\beta$–plane near $\beta = 0$.

*Reference*:


[1] Yu.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 4rd ed., Springer (2023) [Sections 8.4.2 and 8.9]

[2] M. Han, J. Llibre, and J. Yang, On uniqueness of limit cycles in general Bogdanov-Takens bifurcation. *Int. J. Bifurcation & Chaos* **28**(1850115) 12 pp (2018)

# 4 A dual cusp in a sociological model

Consider the planar model

$$\begin{array}{rcrcl} \dot{x} & = & P(x,y) & := & x^2 - x^3 - xy \\ \dot{y} & = & Q(x,y) & := & \beta y^2 - \alpha\beta y^3 - xy \end{array} \tag{4.1}$$

for self-organized segregation.

1. Explain the parameters $\alpha$ and $\beta$ and why (4.1) is a model for segregation.
2. Compute the equilibria of (4.1) and their stability.
3. Derive the bifurcation diagram for (4.1).

*References*:


[1] T.C. Schelling, Dynamic models of segregation. *J. Math. Sociol.* **1**(2):143–186 (1971) [Pages 143–148 & 167*ff*]

[2] D.J. Haw and S.J. Hogan, A dynamical systems model of unorganized segregation. *J. Math. Sociol.* **42**(3):113–127 (2018) [Pages 113–121]

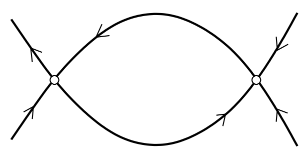
[3] H. Hanßmann and A. Momin, Dynamical systems of self-organized segregation. *J. Math. Sociol.* **48**(3):279–310 (2024) [Sections 1 & 2]

# 5 Bifurcations of monodromic heteroclinic contours in planar systems

1. Discuss bifurcations happening for small $\|\alpha\|$ in generic two-parameter planar systems

$$\dot{X} \;=\; F(X,\alpha) \;, \quad X \in \mathbb{R}^2,\, \alpha \in \mathbb{R}^2, \tag{5.1}$$

   having at $\alpha = 0$ two hyperbolic saddles connected by two heteroclinic orbits forming a monodromic contour:



2. Give an explicit 2D polynomial ODE exhibiting both possible types of such bifurcations.

*References*:


[1] J.W. Reyn, Generation of limit cycles from separatrix polygons in the phase plane. In: *Geometrical Approaches to Differential Equations*. Springer (1980) pp. 264–289

[2] Yu.A. Kuznetsov and J. Hooyman, Bifurcations of heteroclinic contours in two-parameter planar systems: Overview and explicit examples. *Int. J. Bifurcation & Chaos* **31**(2130036) 19 pages (2021)

[3] J. Hooyman, Codim 2 bifurcations of planar heteroclinic contours. Bachelor Thesis. Department of Mathematics, Utrecht University (2020)

# 6 Periodic perturbations of planar Hamiltonian systems

1. Introduce the Melnikov function for a planar Hamiltonian system with a homoclinic orbit to a saddle subject to periodic forcing and discuss its properties.

2. Prove that a simple zero of the Melnikov function implies a transverse intersection of the stable and unstable invariant manifolds of a saddle periodic orbit in the perturbed system, i.e. the existence of a transverse homoclinic orbit to this cycle.

3. Consider the Duffing oscillator with the weak harmonic forcing and damping:

$$\ddot{x} - x + x^3 = \varepsilon(\gamma \cos(\omega t) - \delta \dot{x}) \ , \quad \gamma, \omega, \delta > 0, \ 0 < \varepsilon \ll 1.$$

*References*:


[1] J. Guckenheimer and Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer (1983) [Section 4.5]

[2] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd ed. Springer (2003) [Chapter 28]

# 7 Local stability of periodic orbits

Assume that a smooth system

$$\dot{x} \;=\; f(x) \;\;, \quad x^n \;\in\; \mathbb{R}^n$$

has a periodic solution $\phi(t)$ with the minimal period $\tau_0$.

1. Prove that the Poincaré mapping near the cycle corresponding to $\phi$ is well defined.

2. Establish a relationship between the eigenvalues of the linear part $M$ of the Poincaré mapping and the eigenvalues of the matrix $Y(\tau_0)$, where

$$\dot{Y}(t) \;=\; f_x(\phi(t))Y \;\;, \quad Y(0) = \mathrm{id} \;\;.$$

3. Discuss the notion of the “exponential asymptotic stability with the phase”.

*References*:


[1] J.D. Meiss, *Differential Dynamical Systems.* SIAM (2007) [Sections 4.11 and 4.12]

[2] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials.* [On-line lecture notes, Section 3.2] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect4_5.pdf`

# 8 Period-3 implies chaos

1. Prove the following theorem.

   **Theorem** [Li & Yorke] *Suppose a continuous mapping* $f : [0,1] \longrightarrow [0,1]$ *has a cycle of minimal period* 3*. Then* $f$ *has a cycle of minimal period* $n$ *for all* $n \geq 1$*.*

2. Consider the *logistic mapping*

$$x \mapsto f(x,\alpha) = \alpha x(1-x), \quad x \in [0,1]. \tag{8.1}$$

   Prove that at $\alpha_0 = 1 + 2\sqrt{2}$ the 3rd iterate of (8.1) exhibits a fold bifurcation, generating a stable period 3 cycle and an unstable period 3 cycle as $\alpha$ increases.

*References*:


[1] R.L. Devaney, An Introduction to Chaotic Dynamical Systems. Benjamin/Cummings (1986) [Section 1.10]

[2] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, 2nd ed. CRC Press (1999) [Section 7.3]

[3] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Section 7.1.3] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect13.pdf`

# 9 Lorenz system

1. Explain the origin of the Lorenz-63 ODEs:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \tag{9.1}$$

   where $(\sigma, r, b)$ are positive parameters.

2. Analyse stability of equilibria in (9.1), in particular, derive a condition for a nonzero equilibrium to have a Hopf bifurcation.

3. Prove that the Hopf bifurcation in (9.1) is always subcritical.

4. Describe the sequence of global bifurcations of (9.1) leading to the Lorenz strange attractor and related 1D dynamics.

*References*:


[1] L.P. Shilnikov, A.L.Shilnikov, D.V. Turaev and L. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II. World Scientific (2001) [pp. 877–880]

[2] J. Guckenheimer and Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer (1983) [Sections 6.4 and 5.7]

[3] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Section 6.7 (Exercise E 6.7.3)] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect12.pdf`

[4] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Section 7.6] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect14.pdf`