WISB333; Local Stability of Periodic Orbits in $\mathbb{R}^n$, $n > 2$

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1 Introduction

Periodic orbits play a central role in the qualitative analysis of non-linear dynamical systems. In contrast to equilibria, their stability properties cannot be characterized solely by linearization at a single point, but require tools that capture the dynamics transverse to the orbit. Two such fundamental tools are the Poincaré map and Floquet theory, which reduce the analysis of a continuous-time system to discrete-time dynamics on a transversal section.

The material presented in this essay is largely based on standard results from the theory of non-linear dynamical systems, as developed in classical textbooks and lecture notes, in particular those by Kuznetsov and collaborators. These sources are widely used in both education and research, and many of the proofs and constructions discussed here are well established. Rather than aiming for originality in results, the purpose of this essay is to give a coherent and self-contained exposition of how Poincaré maps, Floquet multipliers, and linearization techniques jointly determine the local stability properties of periodic orbits.

Special emphasis is placed on the notion of *exponential asymptotic stability with phase*. While the condition $|\mu_j| < 1$ for all non-trivial multipliers is often quoted as a stability criterion, its precise interpretation requires care. In particular, the neutral direction corresponding to time translation implies that convergence to a periodic orbit can only be expected up to a phase shift. The aim of this essay is to make this distinction explicit and to show how exponential orbital stability with asymptotic phase follows rigorously from the spectral properties of the Poincaré map.

2 Poincaré maps and Floquet multipliers

The stability analysis of periodic orbits differs in a fundamental way from that of equilibria. While an equilibrium can be studied by linearizing the vector field at a single point, a periodic orbit is an extended object in phase space, and its stability is intrinsically a global-in-time property. Any attempt to characterize its local behaviour must therefore take into account how nearby trajectories evolve over an entire period of the motion.

The Poincaré map provides a robust method for the analysis: by recording successive intersections of trajectories with a transversal section, it encodes the stability of the periodic orbit in a finite-dimensional map. In this way, the study of a periodic solution is reduced to the analysis of a fixed point of a map, bringing the problem closer in spirit to the familiar theory of equilibria. Taking this into account we start by considering smooth n-dimensional system:

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n, \tag{1}$$

with flow $\phi(t, u)$ and a periodic orbit Γ_0 with minimal period τ_0.

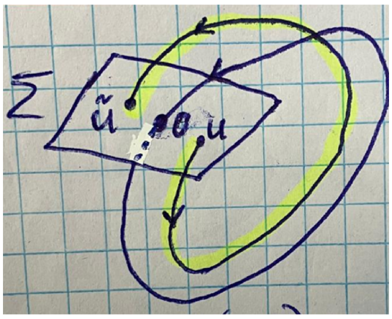


Figure 1: This figure shows an example of a smooth 3-dimensional system with a periodic orbit.

Choose $u_0 \in \Gamma_0$ and define

$$\Sigma_0 := \{\xi \in \mathbb{R}^n \mid \langle f(u_0), \xi \rangle = 0\}.$$

Define a transversal cross-section through u_0 by

$$\Sigma := \{u \in \mathbb{R}^n \mid u = u_0 + \xi,\ \xi \in \Sigma_0\},$$

as shown in Figure 1. Under the identification $u = u_0 + \xi$, the point $\xi = 0$ corresponds to $u = u_0 \in \Gamma_0$. Notice that, by construction, Σ is $(n-1)-$dimensional, which is already a simplification. Of course, if n is large this simplification is minimal. Now, we have the following lemma to assure that the Poincaré map will be well-defined and continuous. We will show this according to the proof in [1].

Lemma 2.1. *There exists a C^1 map $\tau : \Sigma_0 \mapsto \mathbb{R}$, defined in a neighbourhood of $\xi = 0$, such that*

1. $\tau(0) = \tau_0$;
2. $\phi(\tau(\xi), u_0 + \xi) \in \Sigma$.

Moreover, if ξ is sufficiently small and t satisfies

$$\phi(t, u_0 + \xi) \in \Sigma \quad \text{with} \quad |t - \tau_0| \text{ small},$$

then $t = \tau(\xi)$.

Proof. Define $F : \mathbb{R} \times \Sigma_0 \to \mathbb{R}$ by

$$F(t, \xi) = \langle f(u_0), \phi(t, u_0 + \xi) - u_0 \rangle,$$

and consider

$$F(t, \xi) = 0.$$

Then, $F \in C^1$ and

$$F(\tau_0, 0) = \langle f(u_0), u_0 - u_0 \rangle = 0,$$

and

$$\partial_t F(\tau_0, 0) = \langle f(u_0), f(u_0) \rangle = \|f(u_0)\|^2.$$

Note that $f(u_0) \neq 0$, because this is at the periodic orbit, which is not an equilibrium. Therefore,

$$\partial_t F(\tau_0, 0) > 0.$$

Now, the Implicit Function Theorem states that there are open neighbourhood $U \subset \mathbb{R}$ around τ_0 and $V \subset \Sigma_0$ around 0, such that there is a unique C^1-function $\tau : V \to U$ such that

(i) $\tau(0) = \tau_0$

(ii) For all $\xi \in V$:

$$F(\tau(\xi), \xi) = 0$$

(iii) And in particular, if $t \in U$ and $\xi \in V$ such that $F(t, \xi) = 0$ then;

$$t = \tau(\xi) \quad \text{must hold.}$$

This completes the proof. □

Having established the existence, continuity, and the uniqueness of a C^1-map as in the above lemma, we can define the Poincaré map according to the definition below.

Definition 2.2. *The map $P : \Sigma_0 \to \Sigma_0$, defined for $\xi \in \Sigma_0$ near $\xi = 0$ by*

$$P(\xi) = \phi(\tau(\xi), u_0 + \xi) - u_0,$$

*is called the **Poincaré map** associated with the periodic orbit Γ_0.*

Note that there are two closely related Poincaré maps involved. Firstly, the one as defined in Definition 2.2, which satisfies

$$P(0) = 0.$$

And the more well-known Poincaré map

$$\mathcal{P} : \Sigma \to \Sigma := P(u - u_0) + u_0,$$

which is defined on the section Σ and satisfies

$$\mathcal{P}(u_0) = u_0,$$

so that u_0 is a fixed point corresponding to the periodic orbit Γ_0. The two maps $\mathcal{P}$ and P are related by the conjugacy

$$\mathcal{P}(u) = u_0 + P(u - u_0), \qquad u \in \Sigma.$$

Recall that if a system is C^k then its Poincaré map is locally C^k as well. Moreover, we can simply go back in time to derive the inverse of the Poincaré map. Hence, both P and $\mathcal{P}$ are locally well-defined C^n diffeomorphisms. In particular, their inverses are obtained by flowing backward in time. Since we can visualize Σ in the phase-space figure we will consider $\mathcal{P}$ in the figures. However, for the analytical study of stability, it is more convenient to work in coordinates centred at the fixed point u_0. Hence, we analyse the system with P.

Now, to study the local behaviour of the periodic orbit we consider the Taylor expansion of P at $\xi = 0$, where $P(0) = 0$;

$$P(\xi) = P(0) + DP(0) + O(||u\xi||^2) = Mu + O(||u\xi||^2), M \in \mathbb{R}^{(n-1)\times(n-1)}.$$

We can ignore the higher order terms. Thus, from this Taylor expansion we find that $Mu = DP(0)$. The eigenvalues, $\mu_1, \dots, \mu_{n-1} \in \mathbb{C}$, of M are called the multipliers of Γ_0.

2.1 Properties of the multipliers

These multipliers have some important properties , we will discuss these below [2].

2.1.1 Property I

The multipliers are independent of the cross-section Σ and coordinates on Σ. Let us consider two different cross-sections Σ_1 and Σ_2 on Γ_0, as shown in Figure 2a. For each cross-section we can establish a Poincaré map which maps a point on the cross-section to another point along the flow of the system. We call these $\mathbf{P}_1$, with starting point p_1 and end point (after τ_0) $\tilde{p}_1$, and $\mathbf{P}_2$, with starting point p_2 and end point (after τ_0) $\tilde{p}_2$. Then, we can make a mapping Q from Σ_1 to Σ_2 along the flow of the cycle, as shown in Figure 2b. Clearly, Q is continuous and we can trace Q in reverse time which brings us the inverse of Q. Therefore, Q is a local diffeomorphism. In particular, since Q is a diffeomorphism;

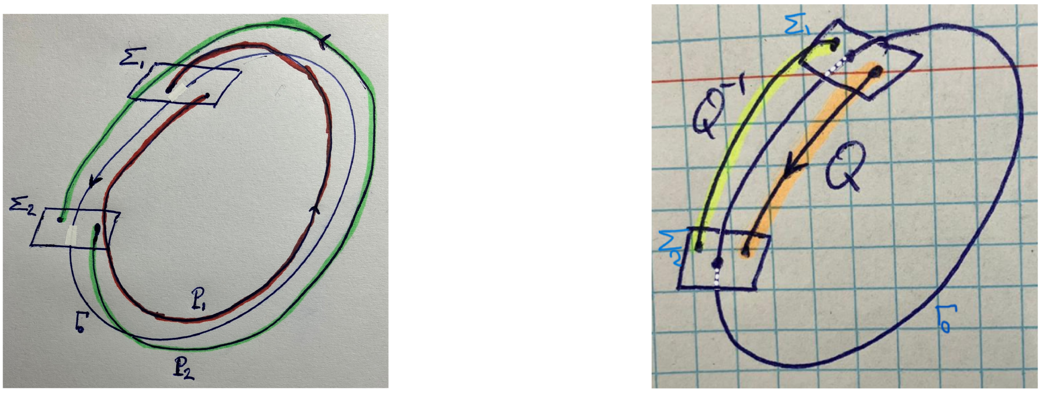
$$\mathbf{P}_2 \circ Q = Q \circ \mathbf{P}_1$$

holds. Hence, $\mathbf{P}_1$ and $\mathbf{P}_2$ are topologically conjugate. This conjugacy implies that the qualitative structures are the same. Moreover, from

$$D\mathbf{P}_2 = DQ\mathbf{P}_1 DQ^{-1}$$

it follows that, even though the linearised matrices might not be identical, the eigenvalues must be the same. This proves the first property of the multipliers.

Figure 2



(a) In this figure the red orbit is $\mathbf{P}_1$ and the green orbit is $\mathbf{P}_2$.

(b) This figure shows an example of how such a Q as mentioned in the text could work.

2.1.2 Property II

We can derive the multipliers via a linearised version of the system.

Lemma 2.3.

$$Y(t) = \left.\frac{\partial \varphi^t(u)}{\partial u}\right|_{u=u_0}$$

satisfies the linear differential equation

$$\dot{Y} = f_u(\varphi^t(u_0))Y$$

and the initial condition $Y(0) = I_n$.

Proof. Let $u(t, u_0 + hv) = \varphi^t(u_0 + hv)$. By definition, for any $v \in \mathbb{R}^n$:

$$[Y(t)]v = \lim_{h\to 0} \frac{1}{h}[u(t, u_0 + hv) - u(t, u_0)].$$

Now;

$$\begin{aligned}[\dot{Y}(t)]v = \frac{d}{dt}[Y(t)]v &= \frac{d}{dt}\lim_{h\to 0} \frac{1}{h}[u(t, u_0 + hv) - u(t, u_0)] \\ &= \lim_{h\to 0} \frac{1}{h}[\dot{u}(t, u_0 + hv) - \dot{u}(t, u_0)].\end{aligned}$$

Recall that, by definition of our system, we have that $\dot{u}(t, u_0 + hv) = f(u(t, u_0 + hv))$. Therefore,

$$= \lim_{h\to 0} \frac{1}{h}[f(u(t, u_0 + hv)) - f(u(t, u_0))].$$

Next, note that if we make a Taylor expansion for f around $u(t, u_0)$ such that

$$f(u(t, u_0 + hv)) = f(u(t, u_0)) + Df(u(t, u_0))[u(t, u_0 + hv) - u(t, u_0)] + o(\|[u(t, u_0 + hv) - u(t, u_0)]\|).$$

Then, we can continue our derivation:

$$\begin{aligned}&= \lim_{h\to 0} \frac{1}{h}[f(u(t, u_0 + hv)) - f(u(t, u_0))] \\ &= f_u(u(t, u_0)) \lim_{h\to 0} \frac{1}{h}[u(t, u_0 + hv) - u(t, u_0)] \\ &= f_u(u(t, u_0))Y(t)v = [f_u(\varphi^t(u_0))Y(t)]v,\end{aligned}$$

for any $v \in \mathbb{R}^n$. Since $\varphi^0(u_0) = u_0$, $Y(0) = I_n$. □

Nota bene that Y depends on u_0 even though we didn't incorporate this into the notation.

Lemma 2.4. *Suppose that* $\varphi^t(u_0)$ *is defined for* $t \in [0, T]$, *where* $T > 0$. *Let* $y_0 = f(u_0)$ *and* $y_1 = f(\varphi^{t_1}(u_0))$, *where* $t_1 \in [0, T]$. *Then* $y_1 = Y(t_1)y_0$.

Proof. Since $u(t) = \varphi^t(u_0)$ is a solution to the Equation 1, we have

$$\frac{d}{dt}\varphi^t(u_0) = f(\varphi^t(u_0)).$$

Differentiating this equation with respect to t we find

$$\frac{d}{dt}\left(\frac{d}{dt}\varphi^t(u_0)\right) = f_u(\varphi^t(u_0))\frac{d}{dt}\varphi^t(u_0),$$

so

$$y(t) = f(\varphi^t(u_0))$$

is a solution to the linearized problem

$$\dot{y} = f_x(\varphi^t(u_0))y, \quad y \in \mathbb{R}^n,$$

with the initial condition $y(0) = f(u_0) = y_0$. Since any such solution to has the form $y(t) = Y(t)y_0$, we get

$$y_1 = y(t_1) = Y(t_1)y_0.$$

□

Let $\Gamma_0 \subset \mathbb{R}^n$ be a periodic orbit (cycle) of the dynamical system generated by Equation 1, i.e. there exists $T > 0$ (the minimal period) such that for every $u_0 \in \Gamma_0$ we have $\varphi^T(u_0) = u_0$ and $\varphi^t(u_0) \neq u_0$ for $t \in (0, T)$. In this case, the orbit $\Gamma_0 = \{u \in \mathbb{R}^n : u = \varphi^t(u_0), 0 \leq t \leq T\}$ is a smooth closed curve.

Theorem 2.5. *For any $u_0 \in \Gamma_0$, $f(u_0)$ is an eigenvector of $Y(T)$ corresponding to eigenvalue 1.*

Proof. By Lemma 2.4, $f(\varphi^T(u_0)) = Y(T)f(u_0)$. The periodicity now yields $f(u_0) = Y(T)f(u_0)$. □

Definition 2.6. *$Y(T)$ is called the **monodromy matrix**. Its eigenvalues are called the (**characteristic** or) **Floquet multipliers**. The multiplier with value 1 is called **trivial**, while all others are called **nontrivial** multipliers.*

Thus, as the second property of the multipliers, we have derived the matrix M via a linearised map. Instead of using the Poincaré map and deriving M via DP. In practice, numerically it is easier to find M via this linearization than via a Poincaré map.

2.1.3 Property III

Finally, via Liouville's formula we derive the following:

$$\det(M) = \mu_1 \cdots \mu_n = \mu_n \exp\left(\int_0^{\tau_0} \operatorname{div}(f(\phi(t, u_0)))dt\right) > 0.$$

Note that via the linearization we have n eigenvalues, instead of $(n-1)$ as we have found via the Poincaré map. This difference is a direct consequence of the transversality of the cross-section on which the Poincaré map works. Therefore, it is common practice to say that $\mu_n = 1$, which is the eigenvalue tangent to the periodic orbit, i.e. the trivial multiplier. This trivial multiplier can easily be derived as follows, choose $v = f(u_0)$, then;

$$Y(\tau_0)v = v.$$

A useful result of this third property is that the product of the multipliers is strictly larger than zero. One might wonder why the multipliers cannot be zero. As mentioned before, the Poincaré map is a local diffeomorphism, which implies that $D\mathcal{P}(0) = M$ is invertible, which means that $\det(D\mathcal{P}(0)) = \det(M) \neq 0$. Note that this property is relatively easy to check numerically. So, when one is analysing a n-dimensional system numerically one should check if the product of the multipliers is strictly greater than zero. If not, something went wrong.

To summarize, as we have seen above, the linearization of the Poincaré map at its fixed point plays a role analogous to the Jacobian matrix at an equilibrium. However, its interpretation requires more care. Whereas the eigenvalues of a Jacobian describe the instantaneous growth or decay of perturbations, the eigenvalues of the linearized Poincaré map measure the net effect of one full revolution around the periodic orbit. They therefore reflect contraction or expansion in directions transverse to the cycle.

This distinction highlights an important conceptual point: stability of a periodic orbit is not a local-in-time notion, but a property over one period. The Floquet multipliers extracted from the linearized Poincaré map provide exactly this information and form the central objects in the stability theory of periodic solutions.

2.2 Exponential asymptotic stability with the phase

Knowing these properties of the multipliers we can continue to analyse the stability of the periodic orbit Γ_0. It starts by noting that when $|\mu_j| < 1$ we call it a stable multiplier, since in this case the orbit convergence to the cycle. And when $|\mu_j| > 1$ we call it an unstable multiplier, because here the close-by orbit diverges, away from the periodic orbit.

Definition 2.7. *A cycle is **hyperbolic** if $|\mu_j| \neq 1$ for $j = 1, 2, 3, \ldots, n-1$.*

As mentioned before $j = n$ means we are considering the eigenvalue tangent to the periodic orbit, which means $\mu_n = 1$ Using this definition we can recall the following theorem, but now for $n-$dimensions.

Theorem 2.8 (Grobman-Hartman). *The Poincaré map $u \mapsto \mathcal{P}(u)$ of a hyperbolic cycle Γ_0 is locally topologically conjugate to its linearization $u \mapsto Mu$.*

Applying the above in a three dimensional scenario we can plot the multipliers as shown in Figure 3 . Define the following eigenspaces:

$$E^s := \{\mu \in M \big| |\mu| < 1\}$$
$$E^c := \{\mu \in M \big| |\mu| = 1\}$$
$$E^u := \{\mu \in M \big| |\mu| > 1\},$$

where $|E^s| = m_s, |E^c| = m_c$, and $|E^u| = m_u$. So $m_s + m_c + m_u = n$. And we define $W^{s,c,u}$ to be the stable, centre, and unstable manifold respectively.

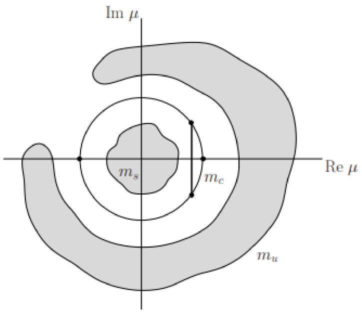


Figure 3: This figure shows an example where the multipliers are compared to the unit circle, copied from [2].

Following the reasoning in [1] we will prove below when a periodic orbit is exponentially asymptotic stable with the phase. And why we can derive this property from its multipliers. An important characteristic of any periodic cycle is its asymptotic stability. Let the nth multiplier (as usual) have value 1, then we have the following definition.

Definition 2.9. *Consider a n-dimensional system such as 1, with a periodic cycle* Γ_0*, then:*

i If all $(n-1)$ *multipliers satisfy* $|\mu| < 1$*, then* Γ_0 *is asymptotically stable.*

ii If $|\mu| > 1$ *for some multiplier, then* Γ_0 *is asymptotically unstable.*

For asymptotically stable periodic cycles the asymptotic phase of a point near an asymptotically stable periodic orbit is the unique phase shift along the orbit such that the trajectory converges exponentially to the corresponding point on the cycle.

Definition 2.10. *A cycle* Γ_0 *through* u_0 *is called* ***exponentially orbitally stable*** *with asymptotic phase if there exist* $c > 0$*,* $K > 1$*, and* $t_0 = t_0(u) \in [0, \tau_0)$ *such that*

$$\|\varphi^t(u) - \varphi^{t-t_0}(u_0)\| \leq Ke^{-ct}, \quad t \geq 0,$$

for all u *with sufficiently small* $\operatorname{dist}(u, \Gamma_0)$.

Next, there is a theorem in [1] which clearly shows how the multipliers determine if the cycle satisfies definition 2.9.

Theorem 2.11. *If all* $(n-1)$ *eigenvalues of the linearization* $\mathcal{P}_\xi(0)$ *of the Poincaré map* $\mathcal{P}$ *at* $\xi = 0$ *satisfy* $|\lambda| < 1$*, then* Γ_0 *is exponentially orbitally stable with asymptotic phase.*

Proof. Consider first $u = u_0 + \xi_0$, $\xi_0 \in \Sigma_0$, and define for a given ξ_0:

$$\begin{aligned} \xi_k &= \mathcal{P}(\xi_{k-1}), \\ \tau_k &= \tau(\xi_{k-1}) + \tau_{k-1}, \ \tau_0 = 0, \end{aligned}$$

for $k = 1, 2, \ldots$. We know that there exists $\delta > 0$ such that for all ξ_0 with $\|\xi_0\| \leq \delta$ the estimate

$$\|\xi_k\| \leq Me^{-\alpha k}\|\xi_0\|$$

holds for some $M \geq 1$ and some $\alpha > 0$. Hence, $\tau(\xi_k) \to T$ and

$$\frac{\tau_k}{kT} \to 1$$

as $k \to +\infty$. But since $\tau \in C^1$, we can derive an explicit estimate:

$$|\tau(\xi_{k-1}) - T| \leq C\|\xi_{k-1}\| \leq MCe^{-\alpha(k-1)}\|\xi_0\|.$$

This implies

$$|(\tau_k - kT) - (\tau_{k-1} - (k-1)T)| = |\tau(\xi_{k-1}) - T| \leq MCe^{-\alpha(k-1)}\|\xi_0\|.$$

From this we derive that $\theta_k = \tau_k - kT$ is a Cauchy sequence, so it has a limit that we denote by t_0. Indeed, we have

$$|\tau_{k+m} - (k+m)T - (\tau_k - kT)| \leq MC\|\xi_0\| \sum_{j=0}^{m-1} e^{-\alpha(k+j)} \leq MC\|\xi_0\| \frac{e^{-\alpha k}}{1-e^{-\alpha}}.$$

Taking the limit $m \to +\infty$, we find

$$|\tau_k - kT - t_0| \leq MC\|\xi_0\| \frac{e^{-\alpha k}}{1-e^{-\alpha}}.$$

Next consider

$$\|\varphi^{t+\tau_k}(u_0+\xi_0) - \varphi^t(u_0)\| = \|\varphi^t(u_0+\xi_k) - \varphi^t(u_0)\| \leq C_1\|\xi_k\| \leq MC_1 e^{-\alpha k}\|\xi_0\| \tag{A}$$

for $0 \leq t \leq T$, since $\varphi^t(x)$ is a C^1-function of (t,x). Likewise

$$\|\varphi^{t+\tau_k}(u_0+\xi_0) - \varphi^{t+kT+t_0}(u_0+\xi_0)\| \leq MC_2 e^{-\alpha k}\|\xi_0\|, \tag{B}$$

for $0 \leq t \leq T$, since the left-hand side is equal to

$$\|\varphi^t(u_0+\xi_k) - \varphi^t(\varphi^{kT+t_0-\tau_k}(u_0+\xi_k))\|.$$

Combining the two inequalities (A) and (B), we find

$$\|\varphi^{t+t_0}(u_0+\xi_0) - \varphi^t(u_0)\| \leq M(C_1+C_2)e^{-\alpha k}\|\xi_0\|$$

for $kT \leq t \leq (k+1)T$.

Now take any $x \in \mathbb{R}^n$ near Γ_0. If this point does not belong to Π_0, consider the first intersection of the forward half-orbit starting at x with Π_0 and represent it as $u_0 + \xi_0$. ☐

2.3 Example

As an example [2], we consider a 3-dimensional system with a hyperbolic cycle in two non-trivial cases, each time one multiplier is stable and the other is unstable. Since the system is 3-dimensional, the cross-section must be 2-dimensional. In one case, both multipliers are positive in the other both are negative, this is a direct consequence of the third property of the multipliers.

In Figure 4 it is shown how to start the analyses by plotting the multipliers on the unit circle. Then, establishing the manifolds on the cross-section. Figure 5 shows how we must visualize our manifolds in 3-dimensional space close to the periodic orbit of the system.

Note that in the positive case, the stable manifolds will remain on the same side while they convergence after each period τ_0 closer to the centre manifold of the periodic cycle. Similarly, the diverging unstable manifolds will remain on one side of the centre manifold. But in the negative case, the (un)stable manifolds will flip with respect to the centre manifold after each period τ_0.

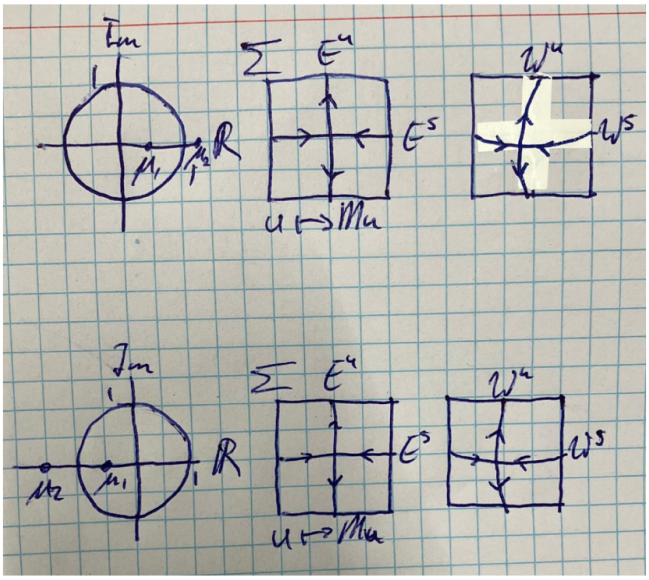


Figure 4: This figure shows how we can start the analysis of the two examples from a plot of the multipliers in $\mathbb{C}$. The upper row shows the exemplary case where both multipliers are positive, the lower row shows the examplary case where both multipliers are negative.

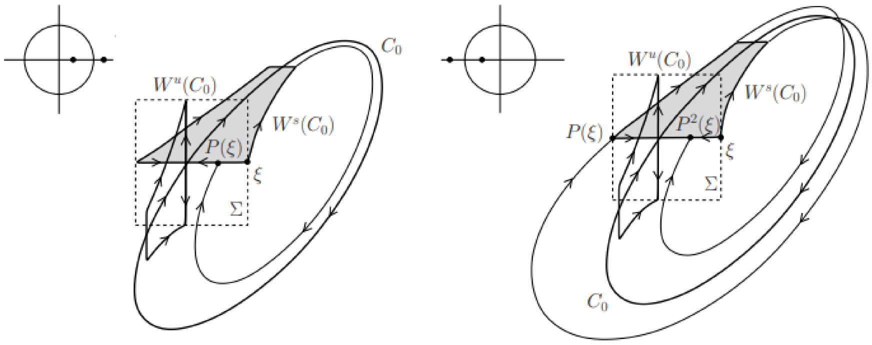


Figure 5: This figure shows the phase portraits of both examples, copied from [2].

3 Conclusion

In this essay we studied the local stability properties of periodic orbits in smooth n-dimensional dynamical systems using Poincaré maps and Floquet theory. By reducing the continuous-time dynamics to a discrete map on a transversal section, the stability problem was translated into a finite-dimensional spectral problem involving the multipliers of the cycle.

We showed that these multipliers can be obtained either from the linearization of the Poincaré map or from the monodromy matrix of the variational equation over one period, and that both approaches lead to the same stability information. In particular, the trivial multiplier equal to one reflects the invariance of the system under time translation and has no analogue in the reduced $(n-1)$-dimensional Poincaré map.

Most importantly, we clarified how the condition $|\mu_j| \neq 1$ for all non-trivial multipliers implies exponential contraction in directions transverse to the periodic orbit, but does not yield convergence to a fixed phase. This leads naturally to the notion of exponential orbital stability with asymptotic phase, which captures the correct stability concept for periodic solutions of autonomous systems. The final theorem makes precise how exponential decay of successive intersections with a transversal section translates into exponential convergence of trajectories to the periodic orbit up to a phase shift.

Thus, the multipliers provide a complete local characterization of the stability of a periodic orbit, provided that their interpretation is made with respect to the intrinsic phase invariance of the system.

References

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