Neutral Saddle Homoclinic Bifurcation in Three Dimensions

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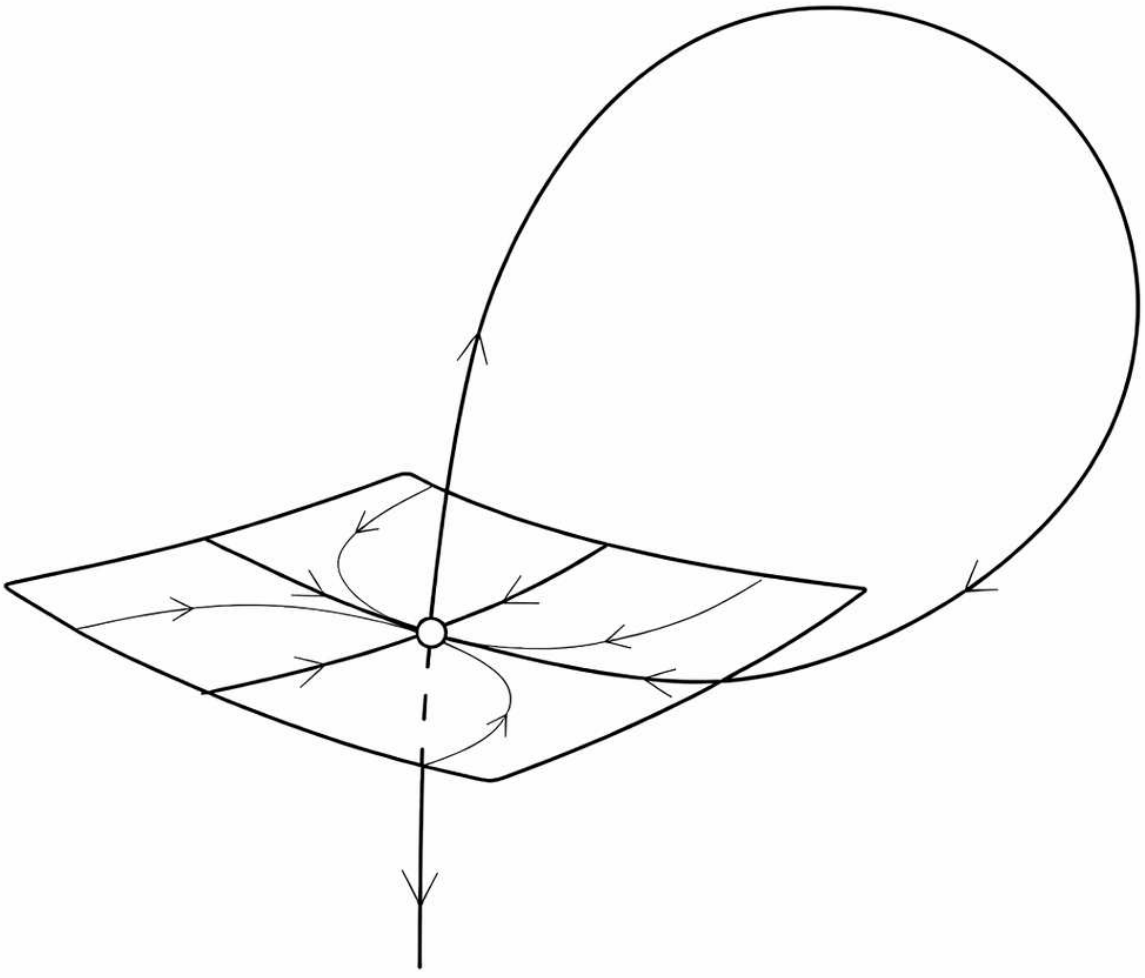

Contents

1 Introduction **3**

2 Describing the problem **4**

2.1 The system 4

2.2 Invariant Manifolds 4

2.3 Homoclinic orbit 5

3 Bifurcation theorems **6**

3.1 Theorem A: non-twisted homoclinic orbits 6

3.2 Theorem B: twisted homoclinic orbits 7

4 Return map **9**

4.1 Construction 9

4.2 Bifurcation analysis 9

5 Example **10**

6 Discussion **11**

1 Introduction

Homoclinic bifurcations play a key role in the dynamical systems theory, as they describe processes of homoclinic orbits and periodic orbits. A homoclinic orbit is a trajectory that leaves an equilibrium along its unstable manifold and returns to that same equilibrium along its stable manifold. The presence of such orbits describe an interaction between the local lineair dynamics of a system and the global structure of its phase space.

In three-dimensional systems homoclinic bifurcations are in particular very interesting, because multiple invariant directions (stable, unstable, strongly stable etc) can interact simultaniously. Results by great minds in this area like Shilnikov show that for example homoclinic orbits to saddle-focus equilibria can generate infinitely many periodic orbits. In this report we will study a different situation: homoclinic orbits to a neutral saddle equilibrium. In our case, we specifically will look at the case where resonance occurs between the leading stable and unstable eigenvalues of the system.

More precisely, we will be looking at a generic three-dimensional system of differential equations depending on two parameters. This system will have a trivial neutral saddle equilibrium and a homoclinic orbit to this equilibrium. Since we work with two parameters, standard methods for analyzing homoclinic bifurcations happening is not enough.

The main objective for this project will be analyzing what kind of bifurcations happen around this neutral saddle equilibrium when our parameters are varied slightly. In particular, I will be explaining why cyclic fold bifurcations of periodic orbits happen, period doubling bifurcations happen and why secondary homoclinic bifurcations emanate from the origin in parameter space, when the homoclinic orbit is twisted. We will use a suitable return map that combines to local dynamics near the equilibrium and the global dynamics along the homoclinic orbit. Furthermore, we will show an explicit example that illustrates how the abstract theory we will cover can be used to determine such bifurcations arising.

Firstly, we will introduce the setting and give the needed assumptions on our system and its properties. Next we will go over some rigorous definitions to understand the abstract theory surrounding this problem. We then will construct the return map and use it to 'prove' a big theorem regarding this problem. From that, we will analyze the bifurcations happening and finally end with an explicit example.

2 Describing the problem

2.1 The system

We will consider a generic three-dimensional system depending on two parameters

$$\dot{u} = F(u, \alpha) \quad , \quad u \in \mathbb{R}^3,\ \alpha \in \mathbb{R}^2 \tag{1}$$

We will give some assumptions one what properties this system needs to have. Firstly, we will make the assumption that the system has an equilibrium $u = 0$ at $\alpha = 0$. At this equilibrium, the linearized system will have some eigenvalues

$$\nu < \mu < 0 < \lambda \quad , \quad \sigma = \mu + \lambda = 0 \tag{2}$$

Since we have both positive and negative real eigenvalues, this equilibrium will be a saddlepoint. Here σ denotes the saddle quantity of our saddlepoint and is determined by the leading stable eigenvalue and leading unstable eigenvalue, which in this case are respectively μ and λ. Since we imposed the *resonance condition*, our saddle quantity is equal to 0. Thus the equilibrium is a *neutral saddle*. Lastly, we impose the condition that the system has a homoclinic orbit Γ_0 to the equilibrium.

2.2 Invariant Manifolds

Since we consider a system with two negative eigenvalues and one positive eigenvalues, the equilibrium is a saddle with two stable directions and one unstable direction. Their associated invariant manifolds play a crucial role in the geometry of the homoclinic orbit. From our eigenvalues, we know that our equilibrium has a *stable manifold* W^s of dimension two. This manifold is tangent to the eigenspaces associated with μ and ν. It consists of all trajectories that approach the equilibrium as the time goes to infinity. Furthermore, the homoclinic orbit returns to the equilibrium along this manifold. On this manifold it looks like a stable node, because we have two negative eigenvalues that correspond to this manifold. Since we have one positive eigenvalue, the equilibrium also has a one-dimensional *unstable manifold* W^u which again is tangent to the eigenspace associated to λ. It consists of all trajectories that approach the equilibrium as time goes to negative infinity. Furthermore, the homoclinic orbit leaves the equilibrium along this manifold. I also want to note that among the stable directions, the eigenvalue ν corresponds to the strongest contraction to the equilibrium. This gives rise to a one-dimensional submanifold of W^s called the *strongly stable manifold* $W^{ss} \subset W^s$. Trajectories that have a component in this manifold will approach the equilibrium very rapidly.

The figure on the front page of this paper gives a good idea on how this situation looks in general, though it does not contain the strongly stable manifold.

2.3 Homoclinic orbit

Now that we have a good notion of the equilibrium, we can take a look at the homoclinic orbit. We will first define it by some parametrisation

$$\Gamma_0 = \{z^*(t) : t \in \mathbb{R}\} \tag{3}$$

such that $z^*(t)$ is a solution to (1) with $\lim_{t\to\pm\infty} z^*(t) = 0$. Next we need some notation to rigourously define a definition that is a key part of this problem. Define

$$e^{\pm} := lim_{t\to\mp\infty} \frac{z(t)}{|z(t)|} \tag{4}$$

Roughly speaking, this defines two unit vectors associated to the homoclinic orbit. One that goes from the unstable manifold to the homoclinic orbit and one that goes from the stable manifold to the homoclinic orbit. We see $e^+ \in T_0W^u$, $e^- \in T_0W^s$, the tangent spaces. We will also define the tangent space to our homoclinic orbit as follows: $T_{z^*(t)} := T_{z^*(t)}W^u + T_{z^*(t)}W^s$. Next we will define some points p, q very close to the equilibrium lying on e^+, e^- respectively. Now we can give a very important definition for our homoclinic orbit.

Def: Let Γ be a homoclinic orbit with real eigenvalues. We call Γ *twisted* if e^- and e^+ point to opposite sides of T_p and T_q. If e^- and e^+ point to the same side of T_p and T_q we will call Γ *nontwisted*.

This definition will let us distinguishe between cases where the space around the homoclinic orbit looks like a band, and cases where the space around the homoclinic orbit looks like a Mobius band (see figure 1).

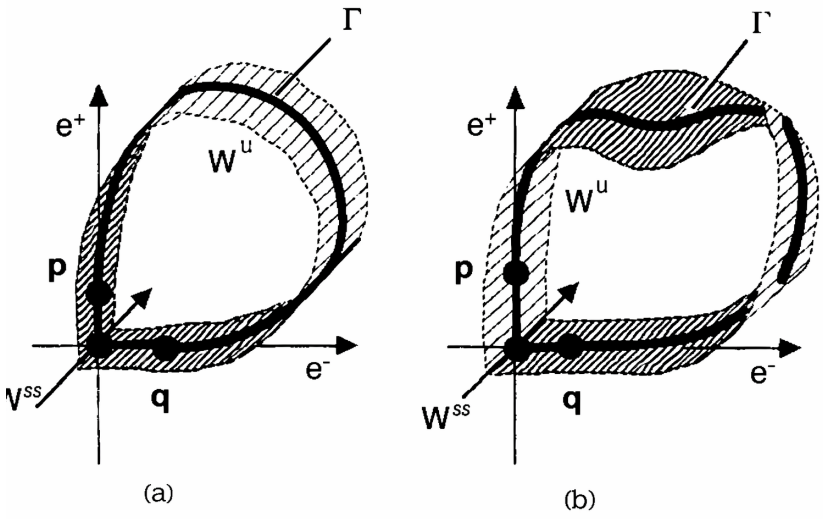

Figure 1: Homoclinic orbits: (a) nontwisted (b) twisted

3 Bifurcation theorems

In this section, my aim is to show via two big theorems what type of bifurcations happen and provide bifurcation diagrams in parameter space. Later on we will discuss how a proof is constructed for these theorems. The last thing we need for this section is the notion of *N-homoclinic orbits and N-periodic orbits.* These are just the respective orbits with winding number N.

3.1 Theorem A: non-twisted homoclinic orbits

Let $F(u, \alpha)$ be a generic two parameter vector field with a nontwisted resonant homoclinic orbit at $\alpha = 0$, then resonant side switching occurs at $\alpha = 0$. Also, there exists a diffeomorphic local change of parameters $\varepsilon = (\varepsilon_1, \varepsilon_2) = \varepsilon(\alpha)$ at $\alpha = 0$ and a function $\varepsilon_2 = \kappa(\varepsilon_1) = 0$ for all $\varepsilon_1 \leq 0$ and $\kappa(\varepsilon_1) > 0$ for all $\varepsilon_1 > 0$, such that the table in figure 2 gives the number of 1-homoclinic- and $1-$periodic orbits.

Region	Definition	No. of 1-hom	No. of 1-per
0	$\varepsilon_1 = 0,\ \varepsilon_2 = 0$	1	0
I	$\varepsilon_1 > 0,\ \varepsilon_2 = 0$	1	1
II	$\varepsilon_1 > 0,\ 0 < \varepsilon_2 < \kappa(\varepsilon_1)$	0	2
III	$\varepsilon_1 > 0,\ \varepsilon_2 = \kappa(\varepsilon_1)$	0	1
IV	$\varepsilon_1 \in \mathbb{R},\ \varepsilon_2 > \kappa(\varepsilon_1)$	0	0
V	$\varepsilon_1 < 0,\ \varepsilon_2 = 0$	1	0
VI	$\varepsilon_2 \in \mathbb{R},\ \varepsilon_2 < 0$	0	1

Figure 2: Resonant side switching

Here resonant side switching means that periodic orbits dissapear from one side of the homoclinic orbit and appear on the other side. In the table there are some region defined by restricting ε. These regions come back in figure 3 viewing the parameter space after our transformation.

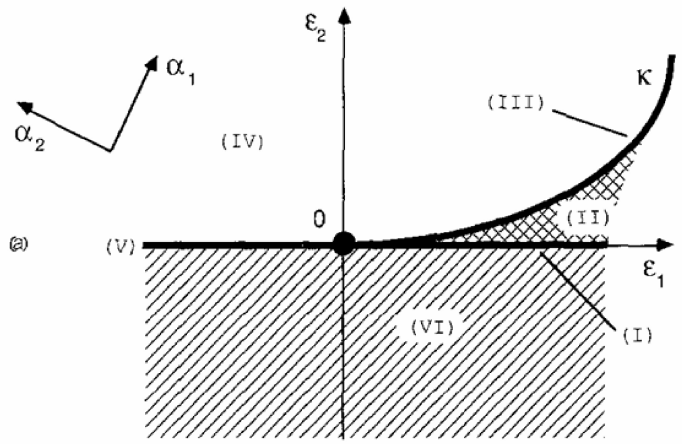

Figure 3: Bifucation diagram for resonant side switching

Analyzing both figures we can determine what kind of bifurcaions arise while varying our parameters. We see if we cross line III while moving from region II to IV, two 1-periodic orbits merge and dissappear at a saddle-node bifurcation. Thus we see that cyclic fold bifurcations happen. Furthermore we can see that crossing line V moving from VI to IV that homoclinic orbits and periodic orbits can suddenly vanish. This type of bifurcation is called a blue sky catastrophe. In figure 4 we see how parameters in each region affect the geometry of the homoclinic orbit. Here, our parameters and regions are labeled differently and the picture is rotated, but it still gives a good intuition.

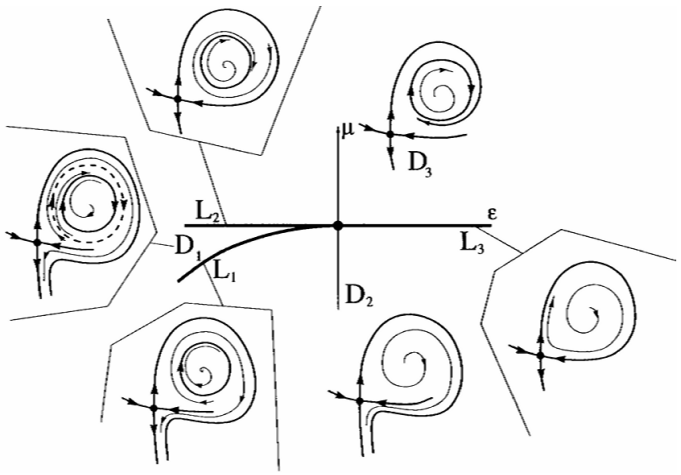

Figure 4: Geometry of nontwisted homoclinic orbit for varied parameters

3.2 Theorem B: twisted homoclinic orbits

Let $F(u, \alpha)$ be a generic two parameter vector field with a twisted resonant homoclinic orbit, then resonant homoclinic doubling occurs. at $\alpha = 0$. Also there exists a diffeomeorphic reparametrization $\varepsilon = \varepsilon(\alpha)$ and two functions $\varepsilon_2 = \kappa_i(\varepsilon_1) = 0$ for $\varepsilon_1 \leq 0$ and $\kappa_i > 0$ for $\varepsilon_1 > 0$ with $i \in \{hom, per\}$, such that the number of N-homoclinic orbits and N-periodic orbits are given by the table in figure 5

Region	Definition	No. of 1-hom	No. of 2-hom	No. of 1-per	No. of 2-per
0	$\varepsilon_1 = 0,\ \varepsilon_2 = 0$	1	0	0	0
I	$\varepsilon_1 > 0,\ \varepsilon_2 = 0$	1	0	0	0
II	$\varepsilon_1 > 0,\ 0 < \varepsilon_2 < \kappa_{\text{per}}(\varepsilon_1)$	0	0	1	$\geqslant 0$
III	$\varepsilon_1 > 0,\ \varepsilon_2 = \kappa_{\text{per}}(\varepsilon_1)$	0	0	1	$\geqslant 0$
IV	$\varepsilon_1 > 0,\ \kappa_{\text{per}}(\varepsilon_1) < \varepsilon_2 < \kappa_{\text{hom}}(\varepsilon_1)$	0	0	1	$\geqslant 1$
V	$\varepsilon_1 > 0,\ \varepsilon_2 = \kappa_{\text{hom}}(\varepsilon_1)$	0	1	1	$\geqslant 0$
VI	$\varepsilon_1 \in \mathbb{R},\ \varepsilon_2 > \kappa_{\text{hom}}(\varepsilon_1)$	0	0	1	$\geqslant 0$
VII	$\varepsilon_1 < 0,\ \varepsilon_2 = 0$	1	0	0	0
VIII	$\varepsilon_1 \in \mathbb{R},\ \varepsilon_2 < 0$	0	0	0	0

Figure 5: Homoclinic doubling

Here homoclinic doubling means doubling the winding number of a homoclinic orbit. Again, in this table there are some regions defined which correspond with different regions of the parameter space given in figure 6. Analyzing both figures we again can figure out what kind of

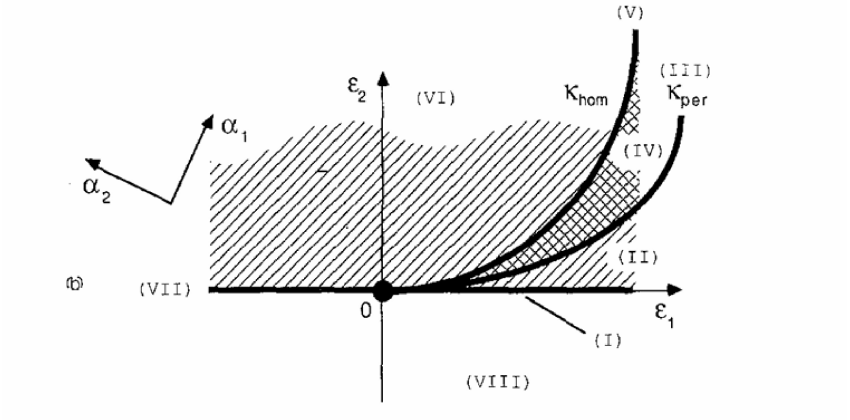

Figure 6: Bifurcation diagram for homoclinic doubling

bifurcations happen. In particular, crossing line III we see that a periodic orbit with a higher period arises. This is thus precisely a period-doubling bifurcation. Again, crossing for example line VII gives rise to a blue sky catastrophe. In figure 7 we see how the twisted homoclinic orbit reacts to these changes in parameters.Again, the labeling is not correct in the picture and it is rotated, but we can still get all the usefull information we need out of it.

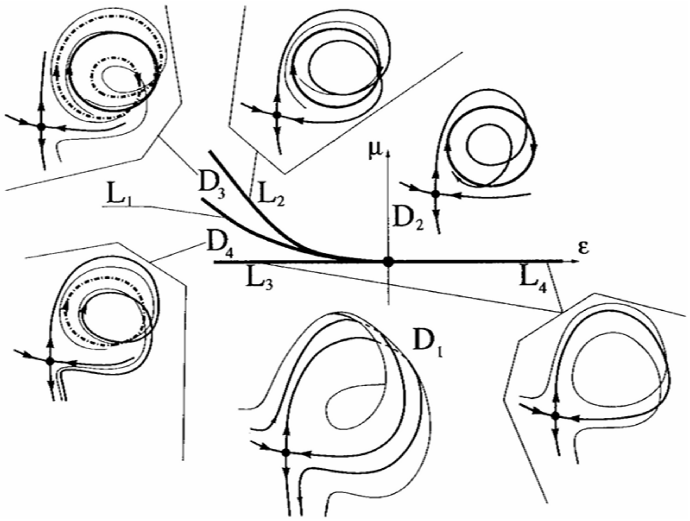

Figure 7: Geometry of twisted homoclinic orbit for varied parameters

4 Return map

4.1 Construction

In this section we will illustrate a basis for a proof of both of the discussed theorems. We will not be giving the entire proof, as that will be too lengthy and difficult for this paper. The proof is based on firstly defining a poincare map $P : \Sigma^- \to \Sigma^+$, where $\Sigma^\pm$ are transverive cross-sections of the homoclinic orbit at p, q respectively. Then switching to a different type of coordinates called Shilnikov coordinates and from there making a reduced form that if easier to work with but describes the dynamics equally well. This leads to the construction of a particular return map. For the entire proof see reference [1]. We will be using the following given return map to analyze bifurcations happening:

$$G(r, \varepsilon) = \varepsilon_2 + a(\varepsilon) r^{1+\varepsilon_1} + h.o.t \tag{5}$$

Here r is a variable introduced by the Shilnikov coordinates and measures the distance to the stable manifold of our homoclinic orbit. Also, $a(\varepsilon)$ is a parameter introduced by making the reduced form. This parameter is either always less then zero (corresponding to a twisted homoclinic orbit) or always bigger then zero (corresponding to a nontwisted orbit).
Now for the second part of the theorems, why such a function ε exists and thus why homoclinic bifurcations emanating from the origin in parameter space exist. We will look at the bifurcation equation $G(r, \varepsilon) = 0$ and directly see that $G(0, 0) = 0$. We note that we are working with a non-degenerate homoclinic orbit, thus $\frac{\partial G}{\partial \varepsilon}(0, 0) \neq 0$. Now by the *Implicit function theorem* we have there exists a smooth function ε such that $G(r, \varepsilon) = 0$. Now that we have illustrated a proof for the theorems, we can analyse the bifurcations happening via the given return map.

4.2 Bifurcation analysis

We will study the fixed point equation

$$G(r, \varepsilon) = r \tag{6}$$

We see that $r = 0$ corresponds with an orbit that lies on the stable manifold, as r measures the distance to the stable manifold. This means that the trajectory leaves the equilibrium along W^u and returns along W^s.Thus $r = 0$ corresponds to homoclinic orbits.

Now for a fixed point $r^* > 0$ we see that the return map corresponds to a trajectory that returns to the cross-section at a positive distance from the stable manifold. Also this trajectory closes after a loop. This thus corresponds to a periodic orbit.

Next, we see that a double root happens for this fixed point equation when $\frac{\partial}{\partial r}G(r,\varepsilon) = 1$. This is exactly the condition that two fixed points of the equation begin to coincide and annihilate eachother. This produces a cyclic fold bifurcation.

Lastly for in particular twisted homoclinic orbits, we recall that period doubling bifurcations happen when the periodic orbit switches stability. This happens exactly when $\frac{\partial}{\partial r}G(r,\varepsilon) = -1$, which of course is easily solveable. In particular this type of bifurcation does not happen when the homoclinic orbit is nontwisted, as $\frac{\partial}{\partial r}G(r,\varepsilon) > 0$ in that case.

5 Example

We will look at the following system:

$$\dot{x} = y, \quad \dot{y} = x - ay - xz, \quad \dot{z} = -bz + x^2 \tag{7}$$

We can directly see that at $a = b = 0$ that the system has an equilibrium at $u := (x, y, z) = (0, 0, 0)$ as all terms are polynomial with al terms of at least degree 1. Linearization at $u = 0$ gives the Jacobian of the system

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -a & 0 \\ 0 & 0 & -b \end{pmatrix}$$

which has eigenvalues $\lambda_\pm = \frac{-a \pm \sqrt{a^2+4}}{2}$ and $\nu = -b$. The origin is thus an saddle. We want to check that our system has all the properties discussed in this project. We want our saddle quantity to be equal to zero, thus $\lambda_- + \lambda_+ = 0$. Solving this gives $a = 0$, and so $\lambda_\pm = \pm 1$. Now let $\lambda_- = \mu$ and $\lambda_+ = \lambda$. Then for the condition that $\nu < \mu < 0 < \lambda$ we impose the condition that $b > 1$. Thus we are in the setting of our project. The onyl thing we still need to show is that there exists a homoclinic orbit to this saddle. To prove this, one can construct a perturbed hamiltonian system, where to unperturbed system corresponds to (7). Then using the Melnikov function, one can prove the existence of a homoclinic orbit. Furthermore, we can construct the homoclinic orbit explicitely by integrating. In this case it turns out that the homoclinic orbit is nontwisted, because it lives completely in the (x, y)-plane. Thus since we meet all the requirements met, we can directly apply Theorem A and can conclude that (a.o) cyclic fold bifurcations happen.

6 Discussion

In this project, we have seen that for general three dimensional systems we only need to impose a few conditions on them to make a lot of progress in understanding the dynamics. In this case in particular, we have used and gave a guidance of proof for two large theorems explaining these dynamics for both nontwisted and twisted homoclinic orbits. This approach that we have used might be useful for other types of three-dimensional systems that impose different starting conditions then ours, to see what kind of bifurcations will arise in those systems.

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