Normal Forms

Seeam Amini
0965243
Inleiding niet lineaire dynamische systemen (WISB333)

1 Introduction

In this essay we look at normal forms for local dynamical systems. The general theme is simple: near an equilibrium (or fixed point) we try to simplify the Taylor expansion as much as we can, without changing the essential behaviour of the system. Results like the Hartman–Grobman theorem already say that hyperbolic equilibria look like their linearization from a topological point of view, but that is not enough for bifurcation problems where eigenvalues lie on the imaginary axis or on the unit circle. In those cases one needs a smoother change of variables to see how nonlinear terms influence phenomena such as limit cycles or invariant circles.

Normal form theory is a systematic way to do this. Using a sequence of near-identity coordinate changes we remove as many nonlinear terms as possible from the vector field or map. The terms that survive, usually called the *resonant terms*, are the ones that actually matter for the local dynamics.

The presentation here roughly follows Kuznetsov [1], Guckenheimer & Holmes [2], and Murdock [3]. First we derive the Poincaré–Dulac normal form for flows, then we apply it to the Hopf and Bogdanov–Takens bifurcations in the plane, and finally we discuss the analogue for discrete-time maps (with multiplicative instead of additive resonance).

1.1 Formal vs. analytic normal forms and non-uniqueness

Normal form transformations are typically constructed as *formal* power series

$$x = y + h_2(y) + h_3(y) + \cdots, \qquad h_k \in \mathcal{H}_k.$$

Formal normalization almost always exists, but convergence is another story because of *small divisors* (the homological denominators can get very small or even hit zero in resonant situations). Analytic normalization usually needs extra spectral or arithmetic assumptions. If the eigenvalues lie in a Poincaré domain (the convex hull of $\lambda_1, \ldots, \lambda_n$ does not contain the origin [5, Ch. 5]), then the divisors $\langle m, \lambda\rangle - \lambda_j$ stay bounded away from zero and convergence follows. When the eigenvalues are on the imaginary axis the divisors can become arbitrarily small; a *Diophantine condition* such as $|\langle m, \lambda\rangle - \lambda_j| \geq C|m|^{-\tau}$ for constants $C > 0$, $\tau > 0$ and all non-resonant m is then needed to control the growth of the normalizing coefficients. In resonant cases the normal form is *not unique*: one has to choose a particular *normal form style*, i.e. a complement to the range of the homological operator at each degree.

Remark 1.1 (A basic convergence remark). In a Poincaré-type spectral domain (for example when all eigenvalues of A lie in the open left half-plane and there are only finitely many resonances), the "divisors" in the equations $\mathcal{L}_A h_k = F_k$ stay bounded away from zero. In that case one can solve the homological equations with uniformly bounded coefficients and the normalizing series converges in a small neighbourhood of the origin. If the spectrum is elliptic and small divisors appear (for instance purely imaginary eigenvalues with many resonances), then in general we only get a formal normal form, and analytic convergence needs extra arithmetic conditions of Diophantine type.

2 Poincaré–Dulac normal form for flows

In this section we construct a formal near-identity transformation that eliminates as many nonlinear terms as possible from a vector field near an equilibrium. The result is the Poincaré–Dulac normal form: a formal power series in which only resonant monomials survive. In practice one truncates this formal series at a finite degree to obtain an approximate (polynomial) normal form whose lower-order terms faithfully capture the local dynamics.

2.1 Preliminaries

Consider a smooth autonomous system in $\mathbb{R}^n$,

$$\dot{x} = f(x), \quad f(0) = 0. \tag{1}$$

Expanding about the origin,

$$\dot{x} = Ax + \sum_{k=2}^{r} F_k(x) + O(|x|^{r+1}), \tag{2}$$

where $A = Df(0)$ and $F_k \in \mathcal{H}_k$, the space of vector-valued homogeneous polynomials of degree k.

We use a multi-index $m = (m_1, \ldots, m_n)$ with $m_i \geq 0$, $|m| = \sum_i m_i$. A basis of $\mathcal{H}_k$ is given by monomial vector fields

$$x^m e_j = x_1^{m_1} \cdots x_n^{m_n} e_j, \qquad |m| = k,$$

with e_j the j-th standard basis vector.

It is standard to apply a linear change of variables putting A into Jordan normal form. In the semisimple (diagonalizable) case over $\mathbb{C}$, we may assume $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ in complex coordinates.

2.2 Near-identity transformations and the homological equation

Assume terms of degree $< k$ have already been simplified. To simplify the degree-k term, consider a near-identity change of coordinates

$$x = y + h_k(y), \qquad h_k \in \mathcal{H}_k. \tag{3}$$

Differentiating gives $\dot{x} = (I + Dh_k(y))\dot{y}$. Substituting $\dot{x} = Ax + F_k(x) + O(|x|^{k+1})$ and re-expanding in y yields

$$(I + Dh_k(y))\dot{y} = A(y + h_k(y)) + F_k(y) + O(|y|^{k+1}).$$

Since $h_k = O(|y|^k)$, we have $Dh_k = O(|y|^{k-1})$. To compute the vector field up to degree k, the truncation

$$(I + Dh_k(y))^{-1} = I - Dh_k(y) + O(|y|^{2k-2})$$

is legitimate because the neglected terms contribute only to degree $\geq k+1$ when multiplied by $Ay + O(|y|^k)$. Collecting the degree-k terms gives the transformed degree-k contribution

$$\tilde{F}_k(y) = F_k(y) + Ah_k(y) - Dh_k(y)\,Ay. \tag{4}$$

Definition 2.1 (Homological operator)**.** The operator $\mathcal{L}_A : \mathcal{H}_k \to \mathcal{H}_k$ is defined by the Lie bracket with the linear vector field

$$\mathcal{L}_A(h_k)(y) := [h_k(y), Ay] = Dh_k(y)\,Ay - Ah_k(y). \tag{5}$$

Thus eliminating F_k reduces to solving the *homological equation*

$$\mathcal{L}_A h_k = F_k. \tag{6}$$

If $F_k \in \mathrm{Im}(\mathcal{L}_A)$ it can be removed; components in a chosen complement to $\mathrm{Im}(\mathcal{L}_A)$ remain in normal form (resonant terms).

2.3 Resonance in the semisimple case

If $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, then for a monomial $y^m e_j$,

$$\mathcal{L}_A(y^m e_j) = \big(\langle m, \lambda\rangle - \lambda_j\big)\, y^m e_j, \qquad \langle m, \lambda\rangle = \sum_{i=1}^{n} m_i \lambda_i. \tag{7}$$

Definition 2.2 (Resonant monomial)**.** A monomial $y^m e_j$ is **resonant** if its homological divisor vanishes

$$\langle m, \lambda\rangle = \lambda_j. \tag{8}$$

2.4 Poincaré–Dulac theorem (formal)

Theorem 2.3 (Poincaré–Dulac normal form (formal))**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^∞ (or analytic) with $f(0) = 0$ and write the Taylor expansion*

$$f(x) = Ax + \sum_{k\geq 2} F_k(x), \qquad F_k \in \mathcal{H}_k,$$

where $A = Df(0)$ (after a linear change of variables) is in Jordan normal form. Then there exists a formal *near-identity transformation (a formal power series)*

$$x = y + h_2(y) + h_3(y) + \cdots, \qquad h_k \in \mathcal{H}_k,$$

such that in the new formal coordinates

$$\dot{y} = Ay + \sum_{k=2}^{\infty} R_k(y), \tag{9}$$

where each $R_k \in \mathcal{H}_k$ lies in a fixed complement N_k of $\mathrm{Im}(\mathcal{L}_A)$ in $\mathcal{H}_k$

$$\mathcal{H}_k = \mathrm{Im}(\mathcal{L}_A) \oplus N_k.$$

In the semisimple case one may choose N_k to be the span of resonant monomials.

Remark 2.4. (i) The theorem is formal in general: the near-identity series is constructed degree by degree but may diverge because of small divisors (see section 1.1).

(ii) In resonant situations the normal form is not unique, because the complement N_k can be chosen in different ways; each choice defines a different *normal form style*.

(iii) In the nonresonant case (no resonant monomials of degree ≥ 2), every F_k lies in $\mathrm{Im}(\mathcal{L}_A)$ and can be removed, so the system is formally conjugate to its linearization $\dot{y} = Ay$.

Proof sketch by induction on the degree

We sketch the standard proof of theorem 2.3 in the formal category. Write

$$f(x) = \sum_{k\geq 2} F_k(x), \qquad F_k \in \mathcal{H}_k,$$

and fix complements $N_k \subset \mathcal{H}_k$ so that $\mathcal{H}_k = \mathrm{Im}(\mathcal{L}_A) \oplus N_k$ for each $k \geq 2$. Let $\pi_k : \mathcal{H}_k \to N_k$ denote the associated projection.

Induction hypothesis. Assume that for some $k \geq 2$ there is a formal near-identity change

$$x = y + h_2(y) + \cdots + h_{k-1}(y), \qquad h_j \in \mathcal{H}_j,$$

such that in the y-coordinates the system is in normal form up to degree $k-1$

$$\dot{y} = Ay + \sum_{j=2}^{k-1} R_j(y) + \widetilde{F}_k(y) + O(|y|^{k+1}),$$

with $R_j \in N_j$ and $\widetilde{F}_k \in \mathcal{H}_k$ arbitrary.

Degree-k step. We now apply an additional near-identity transformation

$$y = z + h_k(z), \qquad h_k \in \mathcal{H}_k,$$

and repeat the calculation from section 2.2. Up to degree k one finds that the new degree-k term is

$$\widetilde{F}_k^{\mathrm{new}} = \widetilde{F}_k - \mathcal{L}_A h_k.$$

We decompose

$$\widetilde{F}_k = \underbrace{(I-\pi_k)\widetilde{F}_k}_{\in \mathrm{Im}(\mathcal{L}_A)} + \underbrace{\pi_k \widetilde{F}_k}_{=:R_k \in N_k}.$$

By definition of the complement there exists a unique $F_k^{\mathrm{rem}} \in \mathrm{Im}(\mathcal{L}_A)$ such that

$$\widetilde{F}_k = F_k^{\mathrm{rem}} + R_k, \qquad R_k \in N_k.$$

Since $F_k^{\mathrm{rem}} \in \mathrm{Im}(\mathcal{L}_A)$, there is $h_k \in \mathcal{H}_k$ solving the homological equation

$$\mathcal{L}_A h_k = F_k^{\mathrm{rem}}.$$

With this choice we obtain

$$\widetilde{F}_k^{\mathrm{new}} = \widetilde{F}_k - \mathcal{L}_A h_k = R_k \in N_k,$$

so all nonresonant degree-k terms are removed and only the prescribed normal-form part R_k remains.

Control of orders. Because $h_k = O(|z|^k)$, the extra terms produced by this degree-k transformation are of order at least $k+1$, so the already normalized degrees $< k$ are unchanged and the remainder remains $O(|z|^{k+1})$. Iterating this construction for $k = 2, 3, \ldots$ yields a formal near-identity series putting the system into normal form to all orders.

3 Applications in $\mathbb{R}^2$

We now apply the general normal form construction from section 2 to two concrete bifurcation problems in the plane. In both cases the system is assumed to be already reduced to a two-dimensional center manifold (see [4, Ch. 3] for the center manifold theorem), so that all eigenvalues of the linearization at the bifurcation point lie on the imaginary axis.

3.1 Hopf bifurcation

Consider a smooth family of planar systems

$$\dot{x} = f(x, \alpha), \qquad x \in \mathbb{R}^2,\ \alpha \in \mathbb{R}, \tag{10}$$

with $f(0, \alpha) = 0$. Suppose that at $\alpha = 0$ the Jacobian $Df(0,0)$ has a complex conjugate pair of eigenvalues $\lambda_{1,2}(0) = \pm i\omega_0$ with $\omega_0 > 0$ and that the eigenvalues cross the imaginary axis with nonzero speed, i.e. the *Hopf transversality condition* holds

$$\frac{d}{d\alpha} \mathrm{Re}\, \lambda_1(\alpha)\Big|_{\alpha=0} \neq 0,$$

where $\lambda_1(\alpha)$ denotes one of the two conjugate eigenvalues of $Df(0, \alpha)$. On the center manifold and after a linear change of variables, the linear part takes the form of a rotation block

$$A(0) = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.$$

Introducing the complex coordinate $z = x_1 + ix_2$ diagonalizes this linear part (the eigenvalues of $A(0)$ are $\pm i\omega_0$), and the reduced system becomes

$$\dot{z} = \lambda(\alpha)z + \sum_{j+k\geq 2} g_{jk}(\alpha)\, z^j \bar{z}^k, \qquad \lambda(\alpha) = \mu(\alpha) + i\omega(\alpha), \tag{11}$$

with $\mu(0) = 0$ and $\omega(0) = \omega_0$.

3.1.1 Resonant cubic term

At $\alpha = 0$ the eigenvalues are $(i\omega_0, -i\omega_0)$. A monomial $z^{m_1}\bar{z}^{m_2}$ in the $\dot{z}$ equation is resonant if

$$m_1(i\omega_0) + m_2(-i\omega_0) = i\omega_0 \quad \Longleftrightarrow \quad m_1 - m_2 = 1.$$

For quadratic terms one would need $m_1 + m_2 = 2$ together with $m_1 - m_2 = 1$, which gives $m_1 = \frac{3}{2}$ and hence no integer solution; therefore all quadratic terms are nonresonant and can be removed. For cubic terms ($m_1 + m_2 = 3$), the unique integer solution of $m_1 - m_2 = 1$ is $(m_1, m_2) = (2, 1)$, corresponding to $z^2\bar{z} = z|z|^2$.

Applying the Poincaré–Dulac procedure from section 2 to (11), one removes all nonresonant terms up to cubic order. The only surviving cubic monomial is the resonant term $z|z|^2$, so the normal form on the center manifold (to cubic order) is

$$\dot{z} = (\beta + i\omega_0)z + c_1 z|z|^2 + O(|z|^4), \tag{12}$$

where $\beta = \mu(\alpha)$ is the real part of the eigenvalue (which vanishes at the Hopf point $\alpha = 0$) and $c_1 \in \mathbb{C}$ is the coefficient of the resonant cubic term, determined by the original nonlinearities.

Writing $z = re^{i\theta}$ gives

$$\dot{r} = \beta r + \mathrm{Re}(c_1)\, r^3 + O(r^5), \tag{13}$$

$$\dot{\theta} = \omega_0 + \mathrm{Im}(c_1)\, r^2 + O(r^4). \tag{14}$$

Definition 3.1 (First Lyapunov coefficient)**.** The (real) first Lyapunov coefficient is the sign-relevant quantity

$$l_1 := \mathrm{Re}(c_1(0)),$$

up to a nonzero scaling depending on normalization of time and parameter.

Assuming nondegeneracy $l_1 \neq 0$, the Hopf bifurcation is:

- **Supercritical** if $l_1 < 0$ (a stable small limit cycle is created as the equilibrium loses stability).
- **Subcritical** if $l_1 > 0$ (an unstable small limit cycle exists on the side where the equilibrium is stable).

3.1.2 Explicit planar formula (coordinate warning)

For a planar system written as

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & -\omega_0\\ \omega_0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}F(x_1,x_2)\\ G(x_1,x_2)\end{pmatrix}, \qquad F, G = O(|x|^2),$$

a classical formula (see [2, Ch. 3]) gives the first Lyapunov coefficient as

$$\begin{aligned} l_1 = \frac{1}{16}\Big(&F_{xxx} + F_{xyy} + G_{xxy} + G_{yyy} \\ &+ \frac{1}{\omega_0}\big[F_{xy}(F_{xx}+F_{yy}) - G_{xy}(G_{xx}+G_{yy}) - F_{xx}G_{xx} + F_{yy}G_{yy}\big]\Big)_{(0,0)}. \end{aligned} \tag{15}$$

Important: (15) assumes the linear part has already been transformed to the rotation block above; applying it in arbitrary coordinates is not valid.

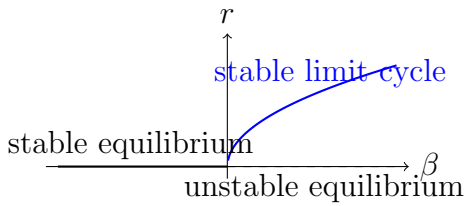

Figure 1: Supercritical Hopf bifurcation: amplitude $r \geq 0$ vs. parameter β.

Figure 1 shows the standard amplitude picture for the supercritical case.

3.2 Bogdanov–Takens bifurcation

The Bogdanov–Takens (BT) bifurcation is a codimension-2 bifurcation with a double zero eigenvalue and a nontrivial Jordan block. After a linear change of variables, the linearization at the BT point is

$$A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}. \tag{16}$$

Because A is non-semisimple (nilpotent), the simple diagonal resonance test from section 2.3 does not apply. In a Jordan basis the homological operator $\mathcal{L}_A$ on homogeneous polynomials is no longer diagonalizable; instead of scalar "divisors" $\langle m, \lambda\rangle - \lambda_j$ one must solve small linear systems coming from the Jordan structure. The choice of complements to $\mathrm{Im}(\mathcal{L}_A)$ is therefore more delicate, and the resulting normal forms are usually presented in a *Takens* style adapted to the nilpotent part rather than as a direct eigenvalue-based resonance test; see [6] for a comprehensive treatment of normal forms for planar vector fields. In practice one works on the (two-dimensional) centre manifold and uses a two-parameter unfolding.

3.2.1 Quadratic BT normal form via the homological operator (sketch)

Write the linear part as the nilpotent vector field

$$\dot{\eta} = A\eta, \qquad A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},$$

so $\dot{\eta}_1 = \eta_2$ and $\dot{\eta}_2 = 0$. Let $F_2 \in \mathcal{H}_2$ be the quadratic part of the (centre-manifold reduced) vector field and write it in the monomial basis as

$$F_2(\eta) = \begin{pmatrix} a_1\eta_1^2 + a_2\eta_1\eta_2 + a_3\eta_2^2 \\ b_1\eta_1^2 + b_2\eta_1\eta_2 + b_3\eta_2^2 \end{pmatrix}.$$

Consider a quadratic near-identity change $\eta = z + h_2(z)$ with

$$h_2(z) = \begin{pmatrix} \alpha_1 z_1^2 + \alpha_2 z_1 z_2 + \alpha_3 z_2^2 \\ \delta_1 z_1^2 + \delta_2 z_1 z_2 + \delta_3 z_2^2 \end{pmatrix} \in \mathcal{H}_2.$$

For this nilpotent A we have $Az = (z_2, 0)^\mathsf{T}$. Applying the definition of $\mathcal{L}_A$ and writing $h_2 = (h_{2,1}, h_{2,2})^\mathsf{T}$ componentwise gives

$$\mathcal{L}_A h_2(z) = Dh_2(z)\, Az - Ah_2(z) = \begin{pmatrix} z_2\, \partial_{z_1} h_{2,1}(z) - h_{2,2}(z) \\ z_2\, \partial_{z_1} h_{2,2}(z) \end{pmatrix}.$$

An explicit computation gives

$$(\mathcal{L}_A h_2)_2(z) = z_2 \partial_{z_1}(\delta_1 z_1^2 + \delta_2 z_1 z_2 + \delta_3 z_2^2) = 2\delta_1 z_1 z_2 + \delta_2 z_2^2,$$

so *no choice of h_2 can generate a z_1^2 term in the second component.* Therefore the monomial $\eta_1^2 e_2$ is not in $\mathrm{Im}(\mathcal{L}_A)$ at degree 2 and cannot be removed by a quadratic change of variables.

To obtain the standard BT style one also fixes the *first* equation to remain $\dot{z}_1 = z_2$ (no quadratic terms in the first component). Writing the transformed quadratic term as

$$\widetilde{F}_2 = F_2 - \mathcal{L}_A h_2,$$

the coefficient of z_1^2 in $(\widetilde{F}_2)_1$ equals $a_1 + \delta_1$ (because $(\mathcal{L}_A h_2)_1$ contains $-\delta_1 z_1^2$). Hence setting $(\widetilde{F}_2)_1 \equiv 0$ forces $\delta_1 = -a_1$. But then the $z_1 z_2$ coefficient in the *second* component becomes

$$(\widetilde{F}_2)_2 : \quad b_2 - 2\delta_1 = b_2 + 2a_1,$$

which is generically nonzero and therefore cannot simultaneously be eliminated while keeping $\dot{z}_1 = z_2$. This is the basic homological-operator reason why, in the BT normal form, one retains both an η_1^2 and a mixed $\eta_1\eta_2$ term. By further rescaling of coordinates and time one can normalize the coefficient of η_1^2 to 1 and the coefficient of $\eta_1\eta_2$ to ± 1; only the sign cannot be removed.

3.2.2 Canonical (topological) BT normal form

For a generic two-parameter family, the reduced dynamics on the center manifold are locally topologically equivalent to

$$\dot{\eta}_1 = \eta_2, \tag{17}$$
$$\dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2, \tag{18}$$

where (β_1, β_2) are (rescaled) unfolding parameters. Under the nondegeneracy conditions, the higher-order remainder $O(|\eta|^3 + |\beta|\,|\eta| + |\beta|^2)$ does not affect the topological type: the local

bifurcation diagram near the BT point is completely determined by the displayed quadratic terms.

The nondegeneracy conditions amount to requiring that, after centre-manifold reduction, the two key quadratic coefficients (corresponding to η_1^2 and $\eta_1\eta_2$ in the second equation) are nonzero, and that the unfolding parameters (β_1, β_2) provide a nondegenerate parametrization of the two-parameter family near the BT point (see [1, Sec. 8.4] for precise statements).

Replacing η_2 by $\dot{\eta}_1$ (using (17)) yields the scalar second-order form

$$\ddot{\eta}_1 = \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\dot{\eta}_1 + \text{higher-order terms.} \tag{19}$$

Note the essential *linear unfolding* term $\beta_2\eta_1$ and the *nonlinear damping* term $\pm\eta_1\dot{\eta}_1$.

3.2.3 Local bifurcation curves (quadratic truncation)

Truncating (17)–(18) at quadratic order,

$$\dot{\eta}_1 = \eta_2, \tag{20}$$
$$\dot{\eta}_2 = \beta_1 + \beta_2\eta_1 + \eta_1^2 \pm \eta_1\eta_2, \tag{21}$$

the equilibria satisfy $\eta_2 = 0$ and $\beta_1 + \beta_2\eta_1 + \eta_1^2 = 0$. The discriminant is $\Delta = \beta_2^2 - 4\beta_1$, and the following bifurcation curves emanate from the BT point.

- **Saddle-node (fold) curve T.** Two equilibria coalesce when the discriminant vanishes:
$$T = \{(\beta_1, \beta_2) : \Delta = 0\} = \{(\beta_1, \beta_2) : 4\beta_1 - \beta_2^2 = 0\}. \tag{22}$$

- **Hopf curve H.** At an equilibrium $(\eta_1^*, 0)$ the linearization of (17)–(18) has trace $\pm\eta_1^*$. A Hopf bifurcation requires the trace to vanish, hence $\eta_1^* = 0$ and $\beta_1 = 0$; additionally $\beta_2 < 0$ is needed for the equilibrium to be a focus:
$$H = \{(\beta_1, \beta_2) : \beta_1 = 0,\ \beta_2 < 0\}. \tag{23}$$

- **Homoclinic curve P.** A Melnikov-type analysis of the truncated system gives the asymptotic relation (see [1, Sec. 8.4]):
$$P = \left\{(\beta_1, \beta_2) : \beta_1 = -\frac{6}{25}\beta_2^2 + o(\beta_2^2),\ \beta_2 < 0\right\}. \tag{24}$$

Thus the BT point organizes nearby saddle-node, Hopf, and homoclinic bifurcations. Figure 2 summarizes the local arrangement of the bifurcation curves.

4 Normal forms for discrete-time maps

The normalization procedure from section 2 extends naturally to discrete-time maps.

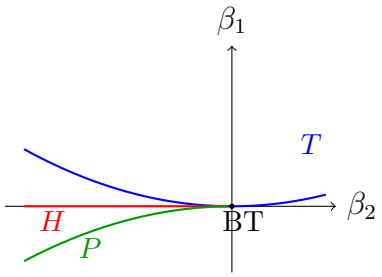

Figure 2: Schematic Bogdanov–Takens bifurcation diagram in the (β_2, β_1)-plane.

4.1 Discrete homological operator

Consider a smooth map $F : \mathbb{R}^n \to \mathbb{R}^n$ with a fixed point at 0,

$$x_{n+1} = F(x_n) = Ax_n + f(x_n), \qquad f(x) = O(|x|^2), \tag{25}$$

where $A = DF(0)$. With a near-identity change of variables $x = y + h(y)$ and expanding order by order, one obtains the discrete homological operator

$$\mathcal{M}_A h_k(y) := h_k(Ay) - Ah_k(y), \tag{26}$$

acting on $h_k \in \mathcal{H}_k$. The degree-k homological equation is $\mathcal{M}_A h_k = F_k$.

4.2 Normal form theorem for maps (formal)

Theorem 4.1 (Poincaré normal form for maps (formal))**.** *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth (or analytic) map with a fixed point at 0 and Taylor expansion*

$$F(x) = Ax + \sum_{k\geq 2} F_k(x), \qquad F_k \in \mathcal{H}_k,$$

where $A = DF(0)$ is in Jordan normal form. Fix for each $k \geq 2$ a complement $N_k \subset \mathcal{H}_k$ so that $\mathcal{H}_k = \mathrm{Im}(\mathcal{M}_A) \oplus N_k$, and let π_k be the associated projection. Then there exists a formal near-identity transformation

$$x = y + h_2(y) + h_3(y) + \cdots, \qquad h_k \in \mathcal{H}_k,$$

such that the conjugated map $G := H^{-1} \circ F \circ H$ with $H(y) = y + h_2(y) + h_3(y) + \cdots$ has the normal form

$$G(y) = Ay + \sum_{k\geq 2} R_k(y),$$

with $R_k \in N_k$ for all $k \geq 2$. In the diagonalizable case one may choose N_k to be the span of multiplicatively resonant monomials.

Proof sketch. The conjugacy equation $H \circ G = F \circ H$ with $H(y) = y + h_2(y) + h_3(y) + \cdots$ is expanded degree-by-degree in homogeneous polynomials. At degree k one obtains a linear equation for $h_k \in \mathcal{H}_k$ of the form

$$\mathcal{M}_A h_k = F_k^{\mathrm{rem}},$$

where F_k^{rem} is the component of the current degree-k term lying in $\mathrm{Im}(\mathcal{M}_A)$ (after previous lower-degree normalizations). Choosing complements $\mathcal{H}_k = \mathrm{Im}(\mathcal{M}_A) \oplus N_k$ and projecting defines the normal-form part $R_k \in N_k$; solving the discrete homological equation eliminates the removable part. Iterating for $k = 2, 3, \ldots$ yields the formal normal form. □

4.3 Multiplicative resonance

If A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$, then for a monomial $y^m e_j$,

$$\mathcal{M}_A(y^m e_j) = \big(\lambda^m - \lambda_j\big)\, y^m e_j, \qquad \lambda^m := \lambda_1^{m_1} \cdots \lambda_n^{m_n}. \tag{27}$$

Thus resonance occurs when $\lambda^m = \lambda_j$, a multiplicative condition (contrast with the additive resonance for flows).

Table 1: Comparison between the homological equations for flows and maps.

	Flows (ODEs)	**Maps**
Homological operator	$\mathcal{L}_A h = Dh(y)Ay - Ah(y)$	$\mathcal{M}_A h = h(Ay) - Ah(y)$
Action on $y^m e_j$	$(\langle m, \lambda\rangle - \lambda_j) y^m e_j$	$(\lambda^m - \lambda_j) y^m e_j$
Resonance condition	$\sum_i m_i \lambda_i = \lambda_j$	$\prod_i \lambda_i^{m_i} = \lambda_j$
Type	additive	multiplicative

Table 1 shows that the structure is essentially the same in both settings, but the resonance condition changes from additive to multiplicative.

4.4 Example: Neimark–Sacker bifurcation

A Neimark–Sacker bifurcation (see [4, Sec. 5.4], [1, Sec. 4.7]) occurs when a complex conjugate pair of eigenvalues crosses the unit circle transversely

$$\lambda_{1,2}(\gamma) = \rho(\gamma) e^{\pm i\theta(\gamma)}, \qquad \rho(0) = 1, \quad \rho'(0) \neq 0, \quad \theta(0) = \theta_0 \in (0, \pi).$$

Assume the *non-strong-resonance* condition

$$e^{ik\theta_0} \neq 1 \quad \text{for } k = 1, 2, 3, 4,$$

which prevents additional resonant monomials of degree ≤ 3 from appearing in the discrete homological equation. At the strong resonances $1:1$, $1:2$, $1:3$, and $1:4$ the normal form contains extra low-order terms and the local dynamics are more complicated.

On the (two-dimensional) center manifold and in a complex coordinate z, the normal form can be written as

$$z_{n+1} = \mu(\gamma) z_n + c_1(\gamma) z_n |z_n|^2 + O(|z_n|^4), \qquad \mu(\gamma) = \rho(\gamma) e^{i\theta(\gamma)}. \tag{28}$$

After rescaling so that $\rho(\gamma) = 1 + \gamma$ to leading order, one commonly writes

$$z_{n+1} = e^{i\theta_0}(1 + \gamma)\, z_n\big(1 + d|z_n|^2\big) + O(|z_n|^4), \qquad d = a + ib \in \mathbb{C}. \tag{29}$$

Writing $z_n = r_n e^{i\phi_n}$ and taking moduli, the factor $|1 + d\, r_n^2| = \sqrt{(1 + ar_n^2)^2 + b^2 r_n^4} = 1 + ar_n^2 + O(r_n^4)$, so the leading radial dynamics are

$$r_{n+1} = (1 + \gamma)\, r_n\, (1 + ar_n^2) + O(r_n^4).$$

An *invariant closed curve* (invariant circle) bifurcates from the fixed point:

- **Supercritical** if $a < 0$ (stable invariant circle for $\gamma > 0$).
- **Subcritical** if $a > 0$ (unstable invariant circle on the opposite side).

Remark 4.2 (Strong resonances). The non-strong-resonance assumption excludes the cases $e^{ik\theta_0} = 1$ for $k = 1, 2, 3, 4$. At such strong resonances (for example $1 : 1$ or $1 : 2$) extra monomials become resonant in the discrete homological equation, which leads to different normal forms and more complicated dynamics; we do not go into these special cases here.

5 Conclusion

Normal form theory reduces local dynamics to essential resonant nonlinearities through the homological equation. For flows the resonance condition is additive in the eigenvalues, whereas for maps it is multiplicative. In the Hopf bifurcation, normal form reduction leaves a single cubic resonant term whose real part controls the creation and stability of a limit cycle. In the Bogdanov–Takens bifurcation, the nilpotent double-zero eigenvalue yields a canonical unfolding with characteristic linear and nonlinear damping terms and organizes saddle-node, Hopf, and homoclinic bifurcations. For maps, the Neimark–Sacker bifurcation produces an invariant circle under standard non-strong-resonance assumptions.

From a practical point of view, the proof of the normal form theorem is also an algorithm: at each degree k one computes the current nonlinear term, projects it onto the complement of $\mathrm{Im}(\mathcal{L}_A)$ to identify the resonant part, and solves a linear system (the homological equation) to determine the coordinate change that removes the rest. In the semisimple case this amounts to reading off the homological divisors $\langle m, \lambda\rangle - \lambda_j$; in the non-semisimple (Jordan) case, as we saw for the Bogdanov–Takens bifurcation, one must additionally respect the Jordan structure, for instance by fixing the first equation $\dot{\eta}_1 = \eta_2$ and solving the remaining linear system for the second component. The algorithm is both constructive and finitely determined at each degree: one needs only linear algebra on the space of homogeneous polynomials.

A natural direction for further study is to go beyond formal normalization and ask when the normalizing transformation actually converges. In Poincaré-type situations (hyperbolic equilibria with finitely many resonances) convergence can often be established [5]; in elliptic problems the small divisors create serious obstructions, and one typically needs Diophantine conditions or must settle for a smooth (but not analytic) normal form via Borel summation. Another interesting question is how the choice of complement N_k (the normal form style) influences the practical usefulness of the result: different styles can lead to different symmetry properties of the normal form, which in turn can simplify the subsequent dynamical analysis.

References

[1] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, 3rd ed., 2004.

[2] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, 1983.

[3] J. Murdock, *Normal Forms and Unfoldings for Local Dynamical Systems*, Springer, 2003.

[4] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, 2nd ed., 2003.

[5] V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, 2nd ed., 1988.

[6] S.-N. Chow, C. Li, D. Wang, *Normal Forms and Bifurcation of Planar Vector Fields*, Cambridge University Press, 1994.