Periodic perturbations of planar Hamiltonian systems

R. Knot
Department of Mathematics, Utrecht University

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I. INTRODUCTION

Hamiltonian systems emerge everywhere in nature, for instance they can represent the total energy. They are relatively easy to study, this is is why they form a good place to start when we want to look at a slightly more complicated situation and perturb the system. The special property of Hamiltonian systems is that they have a potential. In These systems there can appear interesting orbits. One of such orbits are homoclinic orbits. Homoclinic orbits have the special property that they connect a hyperbolic fixed point to itself. So for any point on the homoclinic orbit if you let t go to + or − infinity it goes to this hyperbolic fixed point. It is very natural for a system to dissipate energy. Real oscillators lose mechanical or electrical energy due to effects such as friction and air drag. Damping models this dissipation of energy. In many physical oscillators the motion is not determined solely by the internal restoring forces. The system is continuously acted on by its environment. Forcing is an added external time dependent input to the system. It can make sure the system does not die out due to a damping element. When a system is perturbed a homoclinic orbit can split into two separate orbits, a stable and unstable orbit. This can introduce a transverse homoclinic orbit, which is a signature for chaos. So it is a logical idea to form a measure of distance between the stable and unstable manifold in the prturbed system. The Melnikov function works as a form of measure of the splitting of the manifolds, so as a distance between the two. Int his paper we will study where it comes from and how it is used. We will derive it in 3 steps[1]:

1. Develop a parametrization of the homoclinic "manifold" of the unperturbed system.

2. Develop a measure of splitting of the manifolds in the perturbed system, while using the homoclinic coordinates.

3. Derive the Melnikov function and show its relation to the distace between the manifolds.

II. UNPERTURBED HAMILTONIAN SYSTEMS

Hamiltonian systems in the plane are defined as

$$\begin{cases} \dot{x} = H_y(x,y) \\ \dot{y} = -H_x(x,y) \end{cases} \tag{1}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a smooth Hamiltonian function. We can also write this in vector notation

$$\dot{q} = JDH(q) \tag{2}$$

with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $DH = (H_x, H_y)$A simple example of a Hamiltonian system is shown in Figure 1. A key property of the Hamiltonian function is that it is constant of motion, so $H(x(t), y(t)) = \text{const}$ along solutions of (1)[3]. The class of Hamiltonian systems which we will focus on contains a saddle point and a homoclinic orbit. A homoclinic orbit can be either big or small as seen in Figure 2.

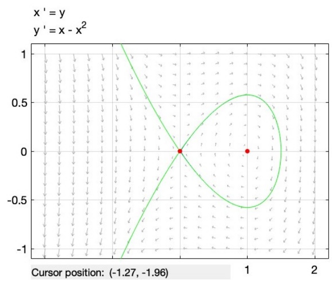

Figure 1. Hamiltonian System with homoclinic orbit to saddle, made with Matlab[4] and pplan9[5]

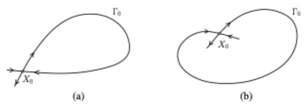

Figure 2. Homoclinic orbits to saddles: “small” (a) and “big” (b)[3]

A hyperbolic saddle equilibrium p_0 always has branches coming into it and going out of it. We define the ones going into the saddle as stable a manifold, $W^s(p_0)$ and the one going out of the saddle as $W^u(p_0)$. So we get a one-dimensional stable manifold

$$W^s(p_0) = \{z : \Phi_t(z) \to p_0 \text{ as } t \to +\infty\},$$

and a one-dimensional unstable manifold

$$W^u(p_0) = \{z : \Phi_t(z) \to p_0 \text{ as } t \to -\infty\},$$

where Φ_t denotes the flow of the ODE. In the plane, each manifold typically has two branches. A homoclinic orbit to p_0 exists if there is a point $z \neq p_0$ such that

$$z \in W^s(p_0) \cap W^u(p_0).$$

Then the entire trajectory $\{\Phi_t(z) : t \in \mathbb{R}\}$ is asymptotic to p_0 both as $t \to +\infty$ and as $t \to -\infty$, which follows from uniqueness of solutions. In particular, if $W^s(p_0)$ and $W^u(p_0)$ coincide along a loop, this loop is a homoclinic orbit.

From now on we will study Hamiltonian systems with two important assumptions:

1. The unperturbed system, with $\varepsilon = 0$, contains a hyperbolic fixed point, p_0 connected to itself with a homoclinic orbit $q_0(t) = (x(t), y(t))$.

2. Let $\Gamma_{p_0} = \{q \in \mathbb{R}^2 : q = q_0(t), t \in \mathbb{R}\} \cup \{p_0\} = W^s(p_0) \cap W^u(p_0) \cup \{p_0\}$, then its interior when we are talking about a small homoclinic orbit, or exterior when we are talking about a big homoclinic orbit is filled with a continuous family of periodic orbits $q^\alpha(t)$ with period $T^\alpha, \alpha \in (-1,0)$. Letting $\lim_{\alpha\to 0} q^\alpha(t) = q_0(t)$ and $\lim_{\alpha\to 0} T^\alpha = \infty$.

When we start with an unperturbed system like (1) and suspend it by adding a differential equation to the system making 3D system like so:

$$\begin{cases} \dot{x} = H_y(x,y) \\ \dot{y} = -H_x(x,y) \\ \dot{\phi} = \omega \end{cases} \tag{3}$$

with $(x, y, \phi) \in \mathbb{R} \times \mathbb{R} \times S^1$ and ω some frequency. This will be useful later on when we perturb the system to make it an autonomous 3D system. Now our hyperbolic saddle equilibrium p_0 becomes a periodic orbit $\gamma(t) = (p_0, \theta(t) = \omega t + \theta_0)$. From this periodic orbit we get a 2-dimensional stable, $W^s(\gamma(t))$ and unstable $W^u(\gamma(t))$ manifold. By our first assumption they coincide and we will call it Γ_γ as shown in Figure 3.

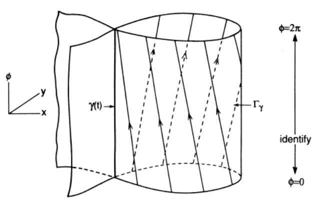

Figure 3. [1]

In the next step we will be studying how Γ_γ breaks under perturbation, but first we note that any point on Γ_γ can be represented as $p = (q_0(-t_0), \phi_0)$ with $t_0 \in \mathbb{R}, \phi_0 \in (0, 2\pi]$. The time t_0 can be interpreted as the time of flight it takes to get to $q_0(0)$ from $q_0(-t_0)$ along the unperturbed homoclinic orbit. Since this flight of time along this homoclinic orbit is unique it follows that the map $(t_0, \phi_0) \mapsto (q_0(-t_0), \phi_0)$ is bijective. This means that we can parametrize $\Gamma_\gamma = \{(q, \phi) : q = q_0(t_0), t_0 \in \mathbb{R}, \phi = \phi_0 \in (0, 2\pi]\}$. For every $p \in \Gamma_\gamma$ we can construct a vector starting at any $p \in \Gamma_\gamma$ in the direction $\pi_p = (DH(q), 0)$, this way it is normal to Γ_γ. We are doing this such that we do not need a perturbed solution later. Notice that both $W^s(\gamma(t))$ and $W^u(\gamma(t))$ intersect π_p transversely as seen in Figure 4. Besides that we can define the cross-section $\Sigma^{\phi_0} = \{(q, \phi) \in \mathbb{R}^2 : \phi = \phi_0\}$ as shown in Figure 5

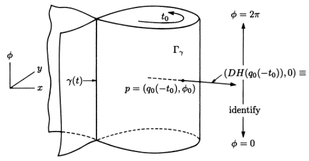

Figure 4. [1]

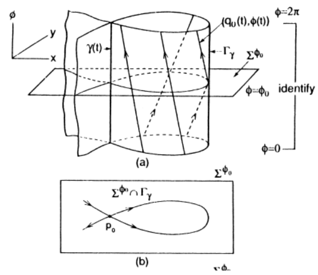

Figure 5. Note that $p_0 = \gamma(t) \cap \Sigma^{\phi_0}$ and $\Gamma_{p_0} = \Gamma_\gamma \cap \Sigma^{\phi_0}$ [1]

III. PERIODIC PERTURBATION

When we perturb a Hamiltonian system it will take the following form:

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon g_1(x,y,t,\varepsilon) \\ \dot{y} = -H_x(x,y) + \varepsilon g_2(x,y,t,\varepsilon) \end{cases} \tag{4}$$

with $(x, y) \in \mathbb{R}^2$, or again in vector notation:

$$\dot{q} = JDH(q) + \varepsilon g(q, t, e) \tag{5}$$

with $g = (g_1, g_2)$. Now for periodic perturbations we assume that g is periodic in t with period $T = 2\pi/\omega$. This is the point where the suspended system becomes useful. When rewrite system (4) as a suspended system like

$$\begin{cases} \dot{x} = H_y(x,y) + \varepsilon g_1(x,y,t,\varepsilon) \\ \dot{y} = -H_x(x,y) + \varepsilon g_2(x,y,t,\varepsilon) \\ \dot{\phi} = \omega \end{cases} \tag{6}$$

with $(x, y, \phi) \in \mathbb{R} \times \mathbb{R} \times S^1$.

Proposition 1. After perturbation we have that, if ε is small enough, the periodic orbit $\gamma(t)$ of the unperturbed (suspended) system persists as a periodic orbit

$$\gamma_\varepsilon(t) = \gamma(t) + \mathcal{O}(\varepsilon),$$

with the same (hyperbolic) stability type as $\gamma(t)$, and $\gamma_\varepsilon(t)$ depends on ε in a C^r manner. Correspondingly, the Poincaré map

$$P_\varepsilon^{\phi_0} : \Sigma^{\phi_0} \to \Sigma^{\phi_0}, \qquad q_\varepsilon(0) \mapsto q_\varepsilon(2\pi/\omega),$$

has a unique hyperbolic saddle point $p_\varepsilon^{\phi_0} = p_0 + \mathcal{O}(\varepsilon)$.

Proof: Define $F(q,\varepsilon) := P_\varepsilon^{\phi_0}(q) - q$. Then $F(p_0, 0) = 0$. This follows from a straightforward application of the implicit function theorem: our assumptions imply that $DP_0^{\phi_0}(p_0)$ does not contain 1 in its spectrum, and hence that $I - DP_0^{\phi_0}(p_0)$ is invertible. Therefore there exists a (unique) smooth curve of fixed points $(p_\varepsilon^{\phi_0}, \varepsilon)$ in $(\Sigma^{\phi_0}, \varepsilon)$ space passing through $(p_0, 0)$, i.e. $P_\varepsilon^{\phi_0}(p_\varepsilon^{\phi_0}) = p_\varepsilon^{\phi_0}$ and $p_\varepsilon^{\phi_0} = p_0 + \mathcal{O}(\varepsilon)$. The corresponding periodic orbit $\gamma_\varepsilon(t)$ inherits hyperbolicity (and hence the same stability type) for ε sufficiently small by continuity. □[2]

Proposition 2. For ε sufficiently small, the local stable and unstable manifolds $W^s_{\text{loc}}(\gamma_\varepsilon(t))$ and $W^u_{\text{loc}}(\gamma_\varepsilon(t))$ exist and are C^r-ε-close to $W^s_{\text{loc}}(\gamma(t))$ and $W^u_{\text{loc}}(\gamma(t))$, respectively (in a fixed neighborhood of γ).

Proof: By Proposition 1, the unperturbed hyperbolic periodic orbit $\gamma(t)$ persists as a hyperbolic periodic orbit $\gamma_\varepsilon(t)$ for ε small. By standard invariant manifold theory for hyperbolic periodic orbits, the local stable and unstable manifolds $W^s_{\text{loc}}(\gamma_\varepsilon(t))$ and $W^u_{\text{loc}}(\gamma_\varepsilon(t))$ exist and depend smoothly (C^r) on ε. In particular, for ε sufficiently small, these local manifolds (and their tangent spaces) are C^r-ε-close to the corresponding unperturbed manifolds in a fixed neighborhood.

Moreover, choosing initial conditions on $W^s_{\text{loc}}(\gamma_\varepsilon)$ (resp. $W^u_{\text{loc}}(\gamma_\varepsilon)$) and comparing with the corresponding unperturbed orbit segment, a standard Grönwall estimate gives $O(\varepsilon)$ closeness on any finite time interval. Once the orbits enter a small neighborhood of γ_ε, exponential contraction forward in time on the stable manifold (resp. backward in time on the unstable manifold) preserves this closeness for all $t \geq 0$ (resp. $t \leq 0$). □ [2] [1]

When we perturb the system it can happen that the stable and unstable manifolds split. This means that the trajectory that would follow the separatrix in the unperturbed system no longer does. The separatrix curve will break into two distinct ones, one that consists of trajectories coming from our saddle and one that consists of the trajectories going into our saddle. This means that a homoclinic orbit is not guaranteed anymore. When the stable and unstable manifolds split we are interested in how far apart they are. This is because if their distance would be zero it implies a homoclinic orbit.

When the perturbation breaks the homoclinic manifold we will get something like Figure 6. For most part we used techniques from Wiggins [1], but Guckenheimer [2] tells us that as long as we work locally near a fixed reference point $p = (q_0(-t_0), \phi_0)$ on the unperturbed homoclinic set. For ε sufficiently small, the local stable and unstable manifolds $W^s_{\text{loc}}(\gamma_\varepsilon)$ and $W^u_{\text{loc}}(\gamma_\varepsilon)$ are C^r graphs in a neighborhood of p. Shrinking the neighborhood if necessary, the normal line π_p intersects each of these local manifolds in a unique point, p_ε^s and p_ε^u. These points can be seen in Figure 6.

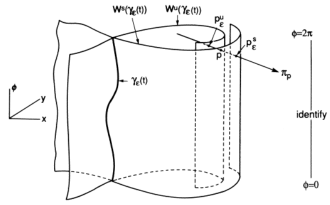

Figure 6. [1]

IV. MELNIKOV FUNCTION

The Melnikov function is used to compute the distance between the stale and unstable manifold. We will do this with the intersection points on the stable and unstable manifolds that we constructed in the last section. First we can define a distance between these points as $d(p,\varepsilon) = |p_\varepsilon^u - p_\varepsilon^s|$. Since both points lie on π_p we can project $p_\varepsilon^u - p_\varepsilon^s$ onto it as follows

$$d(p,\varepsilon) = |p_\varepsilon^u - p_\varepsilon^s| = \frac{(p_\varepsilon^u - p_\varepsilon^s) \cdot (DH(q_0(-t_0)), 0)}{\|DH(q_0(-t_0))\|}.$$

This way we get the signed distance instead of only the magnitude. Now the relative orientation of the stable and unstable manifold near p is respected. Notice that since p_ε^s and p_ε^u lie on π_p they are in the same plane $\phi = \phi_0$, so we can rewrite them as $p_\varepsilon^s = (q_\varepsilon^s, \phi_0)$ and $p_\varepsilon^u = (q_\varepsilon^u, \phi_0)$. Using this construction we can rewrite our distance function such that it is only dependent of t_0, ϕ_0 and ε as follows

$$d(t_0,\phi_0,\varepsilon) = \frac{(q_\varepsilon^u - q_\varepsilon^s) \cdot DH(q_0(-t_0))}{\|DH(q_0(-t_0))\|}. \tag{7}$$

Next we will make a Taylor expansion in $\varepsilon = 0$ this yields the following equation

$$d(t_0,\phi_0,\varepsilon) = d(t_0,\phi_0,0) + \varepsilon \frac{\partial d(t_0,\phi_0,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} + \mathcal{O}(\varepsilon^2). \tag{8}$$

We know that $d(t_0,\phi_0,0) = 0$ since the stable and unstable manifold coincide in the unperturbed system. So we will focus on the second term in this expansion.

$$\frac{\partial d(t_0,\phi_0,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{DH(q_0(-t_0)) \cdot \left(\frac{\partial q_\varepsilon^u}{\partial \varepsilon}\big|_{\varepsilon=0} - \frac{\partial q_\varepsilon^s}{\partial \varepsilon}\big|_{\varepsilon=0}\right)}{\|DH(q_0(-t_0))\|}. \tag{9}$$

In this equation the Melnikov function is defined as

$$M(t_0,\phi_0) = DH(q_0(-t_0)) \cdot \left(\frac{\partial q_\varepsilon^u}{\partial \varepsilon}\Big|_{\varepsilon=0} - \frac{\partial q_\varepsilon^s}{\partial \varepsilon}\Big|_{\varepsilon=0}\right) \tag{10}$$

However, this equation is very hard to actually use because it needs solutions for the perturbed system. Our goal is now to rewrite it without explicit solutions of the unperturbed system. Since $\|DH(q_0(-t_0))\| \neq 0$ on $q_0(-t_0)$ with t_0 finite we have that the Melnikov function is the lowest order nonzero term in the taylor expansion of the distance. Now we want to write the function without the need of the solution of the perturbed vector field. To do this we first construct a time dependent Melnikov function

$$M(t;t_0,\phi_0) = DH(q_0(t-t_0))\cdot\left(\frac{\partial q_\varepsilon^u(t)}{\partial\varepsilon}\Big|_{\varepsilon=0} - \frac{\partial q_\varepsilon^s(t)}{\partial\varepsilon}\Big|_{\varepsilon=0}\right) \quad (11)$$

with $q_\varepsilon^u(0) = q_\varepsilon^u$ and $q_\varepsilon^s(0) = q_\varepsilon^s$ such that $M(0;t_0,\phi_0) = M(t_0,\phi_0)$. Now we will call $\frac{\partial q_\varepsilon^{u,s}(t)}{\partial\varepsilon}\Big|_{\varepsilon=0} = q_1^{u,s}(t)$, where one of u,s can be picked corresponding to the unstable or stable manifold. Now we can write $M(t;t_0,\phi_0) = DH(q_0(t-t_0))\cdot(q_1^u(t) - q_1^s(t))$, and we define:

$$\Delta^{u,s}(t) = DH(q_0(t-t_0))\cdot q_1^{u,s}(t)$$

then it follows that

$$M(t;t_0,\phi_0) = \Delta^u(t) - \Delta^s(t). \quad (12)$$

If we differentiate $\Delta^{u,s}(t)$ w.r.t. t we find

$$\frac{d}{dt}\Delta^{u,s}(t) = \left(\frac{d}{dt}(DH(q_0(t-t_0)))\right)\cdot q_1^{u,s}(t) + DH(q_0(t-t_0))\cdot\frac{d}{dt}q_1^{u,s}(t).$$

Since $q_\varepsilon^{u,s}(t)$ are C^r in ε, t we can change the order of differentiation so we get

$$\frac{d}{dt}\left(\frac{\partial q_\varepsilon^{u,s}(t)}{\partial\varepsilon}\Big|_{\varepsilon=0}\right) = \frac{\partial \dot q_\varepsilon^{u,s}(t)}{\partial\varepsilon}\Big|_{\varepsilon=0}).$$

We know $\dot q_\varepsilon^{u,s}(t) = JDH(q) + \varepsilon g(q,t,e)$ now differentiating with respect to ε and evaluating it to 0 we find

$$\frac{d}{dt}\left(\frac{\partial q_\varepsilon^{u,s}(t)}{\partial\varepsilon}\Big|_{\varepsilon=0}\right) = JD^2H(q_0(t-t_0))\frac{\partial q_\varepsilon^{u,s}(t)}{\partial\varepsilon}\Big|_{\varepsilon=0} + g(q_0(t-t_0,\phi,0)$$

or

$$\frac{d}{dt}q_1^{u,s}(t) = JD^2H(q_0(t-t_0))q_1^{u,s}(t) + g(q_0(t-t_0,\phi,0) \quad (13)$$

which is known as the first variation equation. The solution $q_1^u(t)$ solves the equation for $t\in(-\infty,0]$ and $q_1^s(t)$ solves the equation for $t\in[0,\infty)$. Now if we plug this into the equation for $\frac{d}{dt}\Delta^{u,s}(t)$ we find

$$\begin{aligned}\frac{d}{dt}\Delta^{u,s}(t) = {}& \left(\frac{d}{dt}(DH(q_0(t-t_0)))\right)\cdot q_1^{u,s}(t)\\ &+ DH(q_0(t-t_0))\cdot JD^2H(q_0(t-t_0))q_1^{u,s}(t)\\ &+ DH(q_0(t-t_0))\cdot g(q_0(t-t_0,\phi,0). \quad (14)\end{aligned}$$

The following lemma will simplify this greatly.

Lemma 3. (Lemma 28.1.4 in Wiggins [1])

$$\left(\frac{d}{dt}(DH(q_0(t-t_0)))\right)\cdot q_1^{u,s}(t) + DH(q_0(t-t_0))\cdot JD^2H(q_0(t-t_0))q_1^{u,s}(t) = 0 \quad (15)$$

Proof: First note that

$$\begin{aligned}\frac{d}{dt}\Big(DH\big(q_0(t-t_0)\big)\Big) &= D^2H\big(q_0(t-t_0)\big)\,\dot q_0(t-t_0)\\ &= D^2H\big(q_0(t-t_0)\big)\,JDH\big(q_0(t-t_0)\big).\end{aligned}$$

Let $q_1^{u,s}(t) = (x_1^{u,s}(t), y_1^{u,s}(t))$. Then we have

$$\big(D^2H\big)\big(JDH\big)\cdot q_1^{u,s} = \begin{pmatrix}\frac{\partial^2H}{\partial x^2} & \frac{\partial^2H}{\partial x\partial y}\\ \frac{\partial^2H}{\partial x\partial y} & \frac{\partial^2H}{\partial y^2}\end{pmatrix}\begin{pmatrix}\frac{\partial H}{\partial y}\\ -\frac{\partial H}{\partial x}\end{pmatrix}\cdot\begin{pmatrix}x_1^{u,s}\\ y_1^{u,s}\end{pmatrix}$$

$$= x_1^{u,s}\left(\frac{\partial^2H}{\partial x^2}\frac{\partial H}{\partial y} - \frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial x}\right) + y_1^{u,s}\left(\frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial y} - \frac{\partial^2H}{\partial y^2}\frac{\partial H}{\partial x}\right).$$

Moreover,

$$DH\cdot\big(JD^2H\big)\,q_1^{u,s} = \begin{pmatrix}\frac{\partial H}{\partial x} & \frac{\partial H}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial^2H}{\partial x\partial y} & \frac{\partial^2H}{\partial y^2}\\ -\frac{\partial^2H}{\partial x^2} & -\frac{\partial^2H}{\partial x\partial y}\end{pmatrix}\begin{pmatrix}x_1^{u,s}\\ y_1^{u,s}\end{pmatrix}$$

$$= x_1^{u,s}\left(\frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial x} - \frac{\partial^2H}{\partial x^2}\frac{\partial H}{\partial y}\right) + y_1^{u,s}\left(\frac{\partial^2H}{\partial y^2}\frac{\partial H}{\partial x} - \frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial y}\right).$$

Here we have suppressed the argument $q_0(t-t_0) = (x_0(t-t_0), y_0(t-t_0))$ for notational simplicity. Adding the last two equations gives the result. □

Using this lemma we find that

$$\frac{d}{dt}\Delta^{u,s}(t) = DH(q_0(t-t_0))\cdot g(q_0(t-t_0,\phi,0). \quad (16)$$

Integrating both $\Delta^{u,s}$ from $-\tau$ to 0 and 0 to τ gives

$$\Delta^u(0) - \Delta^u(-\tau) = \int_{-\tau}^{0} DH(q_0(t-t_0))\cdot g(q_0(t-t_0,\omega t+\phi_0,0)\,dt \quad (17)$$

and

$$\Delta^s(\tau) - \Delta^s(0) = \int_{0}^{\tau} DH(q_0(t-t_0))\cdot g(q_0(t-t_0,\omega t+\phi_0,0)\,dt \quad (18)$$

with $\phi = \omega t + \phi_0$. When we add these two and rewrite them we find

$$M(t_0,\phi_0) = M(0;t_0,\phi_0) = \Delta^u(0) - \Delta^s(0) = \int_{-\tau}^{\tau} DH(q_0(t-t_0))\cdot g(q_0(t-t_0,\omega t+\phi_0,0)\,dt - \Delta^u(-\tau) + \Delta^s(\tau). \quad (19)$$

Now we want to consider the limit as τ goes to infinity.
Lemma 4.

$$\lim_{\tau\to\infty}\Delta^u(-\tau)=\lim_{\tau\to\infty}\Delta^s(\tau)=0$$

Proof: As $t\to\infty$ (resp. $-\infty$) $DH(q_0(t-t_0))$ goes to zero exponentially fast, because $q_0(t-t_0)$ approaches a hyperbolic fixed point. Also as $t\to\infty$ (resp. $-\infty$) $q_1^s(t)$ (resp. $q_1^u(t)$) is bounded from Lemma 4.5.2 in Guckenheimer [2]. So both $\Delta^u(-\tau)$ and $\Delta^s(\tau)$ go to zero as τ goes to infinity. □

Lemma 5. (or Lemma 28.1.6 in wiggins [1]) The impropper integral

$$\int_{-\infty}^{\infty} DH(q_0(t-t_0))\cdot g(q_0(t-t_0,\omega t+\phi_0,0)dt \tag{20}$$

converges absolutely.
*Proof:*this follows from the fact that g is bounded for all t and $DH(q_0(t-t_0))$ goes to zero exponentially fast as $t\to\pm\infty$. □ If we shift $t\mapsto t+t_0$ then we find

$$M(t_0,\phi_0)=\int_{-\infty}^{\infty} DH(q_0(t))\cdot g(q_0(t,\omega t+\omega t_0+\phi_0,0)dt. \tag{21}$$

Now it is clear that changing t_0 or ϕ_0 have the same effect on $M(t_0,\phi_0)$and that

$$\frac{\partial M}{\partial\phi_0}(t_0,\phi_0)=\frac{1}{\omega}\frac{\partial M}{\partial t_0}(t_0,\phi_0)$$

Now we will prove that a simple zero of the Melnikov function implies a transverse intersection of the stable and unstable invariant manifolds of a saddle periodic orbit in the perturbed system. Meaning the existence of a transverse homoclinic orbit to this cycle.

Theorem 5. (or 28.1.7 in wiggins [1]) Suppose there exists a point $(t_0,\phi_0)\in\mathbb{R}\times S^1$ such that

1. $M(t_0,\phi_0)=0$, and
2. $\dfrac{\partial M}{\partial t_0}(t_0,\phi_0)\neq 0.$

Then, for ε sufficiently small, $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ intersect transversely at a point

$$\big(q_0(-t_0)+O(\varepsilon),\,\phi_0\big).$$

Proof: Recall that the signed splitting distance satisfies the expansion

$$d(t_0,\phi_0,\varepsilon)=\varepsilon\,\frac{M(t_0,\phi_0)}{\|DH(q_0(-t_0))\|}+O(\varepsilon^2).$$

Define

$$d(t_0,\phi_0,\varepsilon)=\varepsilon\,\tilde d(t_0,\phi_0,\varepsilon),$$

so that

$$\tilde d(t_0,\phi_0,\varepsilon)=\frac{M(t_0,\phi_0)}{\|DH(q_0(-t_0))\|}+O(\varepsilon).$$

Clearly, $\tilde d(t_0,\phi_0,\varepsilon)=0$ implies $d(t_0,\phi_0,\varepsilon)=0$.
At $\varepsilon=0$ we have

$$\tilde d(t_0,\phi_0,0)=\frac{M(t_0,\phi_0)}{\|DH(q_0(-t_0))\|},$$

hence $\tilde d(t_0,\phi_0,0)=0$ since $M(t_0,\phi_0)=0$. Moreover,

$$\frac{\partial\tilde d}{\partial t_0}(t_0,\phi_0,0)=\frac{1}{\|DH(q_0(-t_0))\|}\frac{\partial M}{\partial t_0}(t_0,\phi_0)\neq 0.$$

By the implicit function theorem, there exists a function

$$t_0=t_0(\phi_0,\varepsilon)$$

defined for ε (and $|\phi-\phi_0|$) sufficiently small such that

$$\tilde d(t_0(\phi_0,\varepsilon),\phi_0,\varepsilon)=0,\quad \text{hence}\quad d(t_0(\phi_0,\varepsilon),\phi_0,\varepsilon)=0.$$

Therefore $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ intersect $O(\varepsilon)$-close to $(q_0(-t_0),\phi_0)$.

It remains to show the intersection is transverse. If $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ intersect at a point p, the intersection is transverse if

$$T_pW^s(\gamma_\varepsilon)+T_pW^u(\gamma_\varepsilon)=\mathbb{R}^3.$$

For ε sufficiently small, the points on $W^s(\gamma_\varepsilon)$ and $W^u(\gamma_\varepsilon)$ near p can be parametrized by (t_0,ϕ_0), so

$$\left(\frac{\partial q_\varepsilon^u}{\partial t_0},\frac{\partial q_\varepsilon^u}{\partial\phi_0}\right),\qquad \left(\frac{\partial q_\varepsilon^s}{\partial t_0},\frac{\partial q_\varepsilon^s}{\partial\phi_0}\right)$$

form bases for $T_pW^u(\gamma_\varepsilon)$ and $T_pW^s(\gamma_\varepsilon)$, respectively. The tangent spaces are not equal provided

$$\frac{\partial q_\varepsilon^u}{\partial t_0}-\frac{\partial q_\varepsilon^s}{\partial t_0}\neq 0\quad\text{or}\quad \frac{\partial q_\varepsilon^u}{\partial\phi_0}-\frac{\partial q_\varepsilon^s}{\partial\phi_0}\neq 0.$$

Differentiate $d(t_0,\phi_0,\varepsilon)$ with respect to t_0 and ϕ_0 and evaluate at the intersection point (where $M(t_0,\phi_0)=0$). One obtains

$$\begin{aligned}\frac{\partial d}{\partial t_0}(t_0,\phi_0,\varepsilon)&=\frac{DH(q_0(-t_0))\cdot\left(\frac{\partial q_\varepsilon^u}{\partial t_0}-\frac{\partial q_\varepsilon^s}{\partial t_0}\right)}{\|DH(q_0(-t_0))\|}\\&=\varepsilon\,\frac{\frac{\partial M}{\partial t_0}(t_0,\phi_0)}{\|DH(q_0(-t_0))\|}+O(\varepsilon^2),\\ \frac{\partial d}{\partial\phi_0}(t_0,\phi_0,\varepsilon)&=\frac{DH(q_0(-t_0))\cdot\left(\frac{\partial q_\varepsilon^u}{\partial\phi_0}-\frac{\partial q_\varepsilon^s}{\partial\phi_0}\right)}{\|DH(q_0(-t_0))\|}\\&=\varepsilon\,\frac{\frac{\partial M}{\partial\phi_0}(t_0,\phi_0)}{\|DH(q_0(-t_0))\|}+O(\varepsilon^2).\end{aligned}$$

Hence, if $\frac{\partial M}{\partial t_0}(t_0,\phi_0)\neq 0$ (equivalently $\frac{\partial M}{\partial\phi_0}(t_0,\phi_0)\neq 0$), then for ε sufficiently small at least one of $\partial d/\partial t_0$ or $\partial d/\partial\phi_0$ is nonzero, which implies the manifolds intersect transversely. Using the phase relation

$$\frac{\partial M}{\partial\phi_0}(t_0,\phi_0)=\frac{1}{\omega}\frac{\partial M}{\partial t_0}(t_0,\phi_0),$$

a sufficient condition for transversality is precisely $\frac{\partial M}{\partial t_0}(t_0,\phi_0)\neq 0$.□

V. APPLICATION: DUFFING OSCILLATOR

Now we will be discussing an application of the Melnikov function. Consider the duffing oscillator:

$$\ddot{x} - x + x^3 = \varepsilon(\gamma \cos(\omega t) - \delta \dot{x}) \tag{22}$$

with $\gamma, \omega, \delta > 0, 0 < \varepsilon \ll 1$. We can rewrite this system as a two dimensional system with only first order differential equations as follows

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x^3 + x + \varepsilon(\gamma \cos(\omega t) - \delta \dot{x}) \end{cases} \tag{23}$$

Here the force amplitude γ, frequency ω, and the damping δ are variable parameters and ε is a small scaling parameter.

The unperturbed system has centers at $(x, y) = (\pm 1, 0)$ and a hyperbolic saddle at $(0, 0)$. The separatrix level set is composed of two homoclinic orbits, Γ_0^+ and Γ_0^-, together with the point $p_0 = (0, 0)$:

$$\Gamma_{p_0} = \Gamma_0^+ \cup \Gamma_0^- \cup \{p_0\}.$$

The unperturbed homoclinic orbits based at $q_0(0) = (\pm\sqrt{2}, 0)$ are given by

$$q_0^+(t) = (x_0(t), y_0(t)) = \left(\sqrt{2}\,\mathrm{sech}\, t, -\sqrt{2}\,\mathrm{sech}\, t \tanh t\right),$$
$$q_0^-(t) = -q_0^+(t). \tag{24}$$

We can compute the Melnikov function for q_0^+, the computation for q_0^- is the same. Using the Melnikov integral for the perturbed sys we obtain

$$M_\pm(t_0, \phi_0) = \int_{-\infty}^{\infty} \left[-\delta\big(y^\pm(t)\big)^2 \pm \gamma y^\pm(t) \cos\big(\omega t + \omega t_0 + \phi_0\big)\right] dt. \tag{25}$$

$$M(t_0; \gamma, \delta, \omega) = -\frac{4\delta}{3} + \sqrt{2}\gamma\pi\omega\,\mathrm{sech}(\frac{\pi\omega}{2})\, \sin(\omega t_0).$$

The only term dependent on t_0 or ϕ_0 is $\sin(\omega t_0)$, and we know sin is bounded between -1 and 1. Then it is easy to see that the condition for the manifolds to intersect in terms of the parameters (δ, ω, γ) is given by $\delta < \frac{3\pi \mathrm{sech}\frac{\pi\omega}{2}}{2\sqrt{2}}\gamma$. Poincare maps can be seen in Figure 7 for various parameter values, replace u with x an v with y.

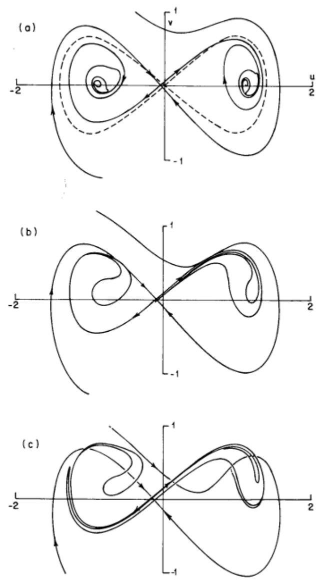

Figure 7. Poincare maps for the Duffing equation showing the stable and unstable manifolds of the saddle near the $(0, 0)$, $\omega = 1.0, \varepsilon\delta = 0.25, (a)\varepsilon\gamma = 0.11, (b)\varepsilon\gamma = 0.19, (c)\varepsilon\gamma = 0.30$ [2]

VI. CONCLUSION

The Melnikov function is an integral used as a measure of the splitting of the stable and unstable manifold. It is the first nonzero term in the taylor approximation of the distance function between points on the manifolds. So it is a first order approximation of the distance between the manifolds in the perturbed system without needing the explicit solutions in the perturbed system. This makes it good to work with because it is generally easier to work withh unperturbed systems than with perturbed ones.

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