INLDS Practicum 10

Study the following two-parameter planar systems exhibiting local codim 2 bifurcations by combining analytical and numerical methods.

Exercises

Ex.1 Noncentral homoclinic orbit to a saddle-node

$$\begin{cases} \dot{x} &= \alpha - x - y + x^2, \\ \dot{y} &= \beta x + \beta y - x - y + \frac{3}{2}x^2 + \frac{3}{2}xy. \end{cases} \tag{1}$$

(a) Verify that at $\alpha = \beta = 0$ the system has a noncentral homoclinic orbit Γ_0 to a saddle-node. *Hint*: The curve $y^2 - x^2(1-x) = 0$ is invariant and Γ_0 is a part of it.

(b) Derive an equation of the saddle-node bifurcation curve passing through the origin of the (α, β)-plane. *Hint*: Look at the equilibrium $x = y = 0$. Construct the phase portraits of (1) along this curve with `pplane9`.

(c) Use `pplane9` to produce all representative phase portraits of the system (1) near the codim 2 bifurcation point $\alpha = \beta = 0$. *Hint*: Consider the phase region $(x, y) \in [-0.5, 1.5] \times [-0.5, 0.5]$ and take $(\alpha, \beta) = (0.1, 0), (-0.1, 0), (-0.1, 0.2)$.

(d) Find by simulations the values α_{Hom} corresponding to the *saddle homoclinic bifurcation* for fixed $\beta = 0.2, 0.15, 0.1,$ and 0.05. Sketch the bifurcation diagram of the system (1) in the parameter region $(\alpha, \beta) \in [-0.25, 0.25] \times [-0.25, 0.25]$.

Ex.2 Neutral saddle homoclinic orbit

$$\begin{cases} \dot{x} &= y + \alpha x + \beta y - \alpha x^2 + \gamma H(x,y) H_x(x,y), \\ \dot{y} &= x - \frac{3}{2}x^2 + \alpha y - \frac{3}{2}\alpha xy + \gamma H(x,y) H_y(x,y), \end{cases} \tag{2}$$

where $H(x,y) = \frac{1}{2}[y^2 - x^2(1-x)]$ and (α, β) are parameters, while γ is a constant.

(a) Verify that for $\alpha = \beta = \gamma = 0$ the system is Hamiltonian and has a neutral saddle homoclinic orbit Γ_0 in the level set $H(x,y) = 0$. Check that the saddle quantity $\sigma = 0$ and that $\int_{-\infty}^{\infty} \text{div } F(X^0(t))dt = 0$, where F is the RHS of (2) and X^0 is the corresponding homoclinic solution.

(b) Prove that for $\alpha = \beta = 0$ and any $\gamma < 0$ the system (2) still has the same neutral saddle homoclinic orbit Γ_0 but now it is stable (from inside), since $\int_{-\infty}^{\infty} \text{div } F(X^0(t))dt < 0$.

(c) Prove that for $\beta = 0, \gamma < 0$, and any $\alpha \neq 0, 3$ the system (2) has the same saddle homoclinic orbit Γ_0 but with $\sigma \neq 0$. Use `pplane9` to illustrate this for $\beta = 0, \gamma = -1$, and $\alpha = \pm 0.3$. Which bifurcations can be expected near these values of α under the variation of β ?

(d) Use `pplane9` to produce all representative phase portraits of (2) near the codim 2 bifurcation point $\alpha = \beta = 0$ for $\gamma = -1$. *Hint*: The most interesting phase portrait (with two cycles) can be seen, for example, at $(\alpha, \beta) = (0.3, 0.15)$.

(e) For $\gamma = -1$, find by simulations the values β_{LPC} corresponding to the *fold bifurcation* (*collision*) *of cycles* for fixed $\alpha = 0.3, 0.2$ and 0.1. Sketch the bifurcation diagram of the system (2) in the parameter region $(\alpha, \beta) \in [-0.3, 0.3] \times [-0.2, 0.2]$.

Homework

Hand-in is exercise 2.