

## INLDS Practicum 9

Study the following two-parameter planar systems exhibiting local codim 2 bifurcations by combining analytical and numerical methods.

### Exercises

#### Ex.1 Takens normal form

$$\begin{cases} \dot{x} &= \beta x + y + x^2, \\ \dot{y} &= \alpha - 4x^2. \end{cases} \quad (1)$$

- Derive equations for the fold and Hopf bifurcations in the system.
- Prove that a Bogdanov–Takens (BT) bifurcation occurs in the system and find the corresponding parameter values.
- Compute the normal form coefficients  $a$  and  $b$  for the BT-bifurcation and verify that  $ab \neq 0$ .
- Use `pp1ane9` to produce all representative phase portraits of the system near the Bogdanov–Takens point. Sketch the bifurcation set of the system and give phase portraits.
- For  $\alpha = 0.25$  find numerically the value of  $\beta$  corresponding to the saddle homoclinic bifurcation.

#### Ex.2 A prey-predator model by Bazykin and Khibnik

$$\begin{cases} \dot{x} &= \frac{x^2(1-x)}{n+x} - xy, \\ \dot{y} &= -y(m-x), \end{cases} \quad (2)$$

where  $x, y \geq 0$  and  $0 < m, n < 1$ .

- Using `pp1ane9`, produce several phase portraits of model (2) for different values of  $m$  and fixed  $n = \frac{1}{4}$  and  $n = \frac{1}{16}$ . *Hint:* The most interesting phase portrait occurs at  $(m, n) = (\frac{1}{5}, \frac{1}{16})$ .
- For  $m = \frac{1}{5}$  approximate by simulations the value of  $n$  corresponding to the collision and disappearance of two periodic orbits.
- Derive an equation for the Hopf bifurcation in the model. *Hint:* Consider the orbitally equivalent (why?) polynomial system

$$\begin{cases} \dot{x} &= x^2(1-x) - xy(n+x), \\ \dot{y} &= -y(m-x)(n+x). \end{cases} \quad (3)$$

- Compute the 1st Lyapunov coefficient  $l_1$  along the Hopf bifurcation line in (3) and prove that this coefficient vanishes at  $(m, n) = (\frac{1}{4}, \frac{1}{8})$ . Conclude that a Bautin bifurcation happens in the model.
- Sketch the bifurcation set of the model and give phase portraits.
- Challenge:** Prove that the Bautin bifurcation is nondegenerate by computing the corresponding 2nd Lyapunov coefficient  $l_2$ .

### Homework

Hand-in is exercise 2.