# 1 Poincaré–Bendixson Theorem

1. Give the definition of the $\omega$–limit sets for bounded orbits of smooth $n$–dimensional ODEs and establish their basic properties.

2. Characterise the $\omega$–limit sets of bounded orbits of smooth planar ODEs, i.e. prove the Poincaré–Bendixson theorem in $\mathbb{R}^2$.

3. Give examples that were not used in the lecture notes or during the practicum.

*References*:


[1] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Chapter 4] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect6_7.pdf`

[2] F. Verhulst, *Nonlinear Differntial Equations and Dynamical Sysyems*, 2nd ed. Springer (1996) [Section 4.2]

[3] J.D. Meiss, *Differential Dynamical Systems*. SIAM (2007) [Sections 4.9, 6.6, and 6.7]

## 2 Study of planar systems at infinity

1. Describe the Poincaré compactification of the plane.

2. For a planar polynomial system, derive the differential equations that define the phase portraits near the border of the Poincaré disk in appropriate coordinates.

3. Discuss the blow-up techniques to study degenerate equilibria that often appear at the border.

4. Prove that the planar system

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -x + h(1 - y^2)y \end{cases} \tag{2.1}$$

   has at least one stable limit cycle for all $h > 0$. *Hint*: Show that for $h > 0$ the origin $x = y = 0$ is an unstable equilibrium of (2.1), while the border of the Poincaré disk is also unstable.

*References*:


[1] J.D. Meiss, *Differential Dynamical Systems*. SIAM (2007) [Section 6.8]

[2] F. Dumortier, J. Llibre, and J. Artés, *Qualitative Theory of Planar Differntial Systems*. Springer (2006) [Chapters 5 and 3]

[3] L. Perko, *Differntial Equations and Dynamical Systems*, 3rd ed. Springer (2001) [Section 3.10]

# 3 Normal forms

1. Prove the Poincaré Normal Form Theorem for ODEs.
2. Apply the theorem to derive the critical normal forms for the Hopf and Bogdanov–Takens bifurcations in $\mathbb{R}^2$.
3. Discuss a variant of the Normal Form Theorem for mappings.

*References*:


[1] Yu.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 4rd ed., Springer (2023) [Section 3.8]

[2] J. Guckenheimer and Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer (1983) [Sections 3.3 and 7.2]

[3] J. Murdock, *Normal Forms and Unfoldings for Local Dynamical Systems*. Springer (2003) [Chapter 1]

[4] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd ed. Springer (2003) [Chapter 19]

# 4 A dual cusp in a sociological model

Consider the planar model

$$\begin{array}{rcrcl} \dot{x} & = & P(x,y) & := & x^2 - x^3 - xy \\ \dot{y} & = & Q(x,y) & := & \beta y^2 - \alpha\beta y^3 - xy \end{array} \tag{4.1}$$

for self-organized segregation.

1. Explain the parameters $\alpha$ and $\beta$ and why (4.1) is a model for segregation.
2. Compute the equilibria of (4.1) and their stability.
3. Derive the bifurcation diagram for (4.1).

*References*:


[1] T.C. Schelling, Dynamic models of segregation. *J. Math. Sociol.* **1**(2):143–186 (1971) [Pages 143–148 & 167*ff*]

[2] D.J. Haw and S.J. Hogan, A dynamical systems model of unorganized segregation. *J. Math. Sociol.* **42**(3):113–127 (2018) [Pages 113–121]

[3] H. Hanßmann and A. Momin, Dynamical systems of self-organized segregation. *J. Math. Sociol.* **48**(3):279–310 (2024) [Sections 1 & 2]

## 5 Uniqueness of the limit cycle near the Bogdanov–Takens bifurcation

Consider the Bogdanov normal form for the Bogdanov–Takens bifurcation

$$\left\{\begin{array}{lcl}\dot{\xi}_1 & = & \xi_2,\\ \dot{\xi}_2 & = & \beta_1+\beta_2\xi_1+\xi_1^2-\xi_1\xi_2.\end{array}\right. \tag{5.1}$$

1. Show that for $\beta_2^2 > 4\beta_1$ system (5.1) is orbitally equivalent to a perturbed Hamiltonian system

   $$\left\{\begin{array}{lcl}\dot{\zeta}_1 & = & \zeta_2\\ \dot{\zeta}_2 & = & \zeta_1(\zeta_1-1)\ -\ (\gamma_1\zeta_2+\gamma_2\zeta_1\zeta_2)\end{array}\right. , \tag{5.2}$$

   where $\gamma_j = \gamma_j(\beta) \to 0$ as $\beta \to 0$.

2. Study periodic and saddle homoclinic orbits in (5.2). In particular, prove that it has exactly one cycle for small $\|\gamma\|$ between the vertical half-axis

   $$H \ = \ \{\gamma : \gamma_1 = 0, \gamma_2 \geq 0\}$$

   and a curve

   $$P \ = \ \{\gamma : \gamma_1 = -\frac{1}{7}\gamma_2 + o(|\gamma_2|), \gamma_2 \geq 0\}\ .$$

3. Discuss the complete bifurcation diagram of the original system (5.1) in the $\beta$–plane near $\beta = 0$.

*Reference*:


[1] Yu.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 4rd ed., Springer (2023) [Sections 8.4.2 and 8.9]

[2] M. Han, J. Llibre, and J. Yang, On uniqueness of limit cycles in general Bogdanov-Takens bifurcation. *Int. J. Bifurcation & Chaos* **28**(1850115) 12 pp (2018)

# 6 Local stability of periodic orbits

Assume that a smooth system

$$\dot{u} \;=\; f(u) \;\;, \quad u \;\in\; \mathbb{R}^n$$

has a periodic solution $\phi(t)$ with the minimal period $\tau_0$.

1. Prove that the Poincaré mapping near the cycle corresponding to $\phi$ is well defined.

2. Establish a relationship between the eigenvalues of the linear part $M$ of the Poincaré mapping and the eigenvalues of the matrix $Y(\tau_0)$, where

$$\dot{Y}(t) \;=\; Df(\phi(t))Y \;\;, \quad Y(0) = \mathrm{id} \;\;.$$

3. Discuss the notion of the "exponential asymptotic stability with the phase".

*References*:


[1] J.D. Meiss, *Differential Dynamical Systems.* SIAM (2007) [Sections 4.11 and 4.12]

[2] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials.* [On-line lecture notes, Section 3.2] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect4_5.pdf`

# 7 Period-doubling bifurcation

1. Describe the period-doubling bifurcation of a fixed point in a smooth scalar mapping

$$x \mapsto f(x,\alpha) \ , \quad x \in \mathbb{R},\, \alpha \in \mathbb{R}. \tag{7.1}$$

2. Prove that in a generic mapping (7.1) a unique 2–cycle appears at this bifurcation, while the fixed point changes stability. Formulate all relevant genericity conditions explicitly.

3. Discuss topological equivalence of a generic mapping (7.1) near the period-doubling bifurcation to the normal form

$$\xi \mapsto -(1+\beta)\xi \pm \xi^3 \ . \tag{7.2}$$

4. Analyse the period doubling bifurcation(s) in the logistic mapping $x \mapsto \alpha x(1-x)$.

*References*:


[1] R.L. Devaney, An Introduction to Chaotic Dynamical Systems. Benjamin/Cummings (1986) [Section 1.12]

[2] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, 2nd ed. CRC Press (1999) [Section 7.3]

[3] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Sections 5.3.2 and 7.1.3]
`http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect10_11.pdf`
`http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect13.pdf`

# 8 Period-3 implies chaos

1. Prove the following theorem.

   **Theorem** [Li & Yorke] *Suppose a continuous mapping* $f : [0,1] \longrightarrow [0,1]$ *has a cycle of minimal period* $3$*. Then* $f$ *has a cycle of minimal period* $n$ *for all* $n \geq 1$*.*

2. Consider the *logistic mapping*

$$x \mapsto f(x, \alpha) = \alpha x(1-x), \quad x \in [0,1]. \tag{8.1}$$

   Prove that at $\alpha_0 = 1 + 2\sqrt{2}$ the 3rd iterate of (8.1) exhibits a fold bifurcation, generating a stable period 3 cycle and an unstable period 3 cycle as $\alpha$ increases.

*References*:


[1] R.L. Devaney, An Introduction to Chaotic Dynamical Systems. Benjamin/Cummings (1986) [Section 1.10]

[2] C. Robinson, Dynamical Systems: Stability, Symbolic Dynamics, and Chaos, 2nd ed. CRC Press (1999) [Section 7.3]

[3] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Section 7.1.3] `http://www.staff.science.uu.nl/∼kouzn101/NLDV/Lect13.pdf`

# 9 Periodic perturbations of planar Hamiltonian systems

1. Introduce the Melnikov function for a planar Hamiltonian system with a homoclinic orbit to a saddle subject to periodic forcing and discuss its properties.

2. Prove that a simple zero of the Melnikov function implies a transverse intersection of the stable and unstable invariant manifolds of a saddle periodic orbit in the perturbed system, i.e. the existence of a transverse homoclinic orbit to this cycle.

3. Consider the Duffing oscillator with the weak harmonic forcing and damping:

$$\ddot{x} - x + x^3 = \varepsilon(\gamma \cos(\omega t) - \delta \dot{x}) \ , \quad \gamma, \omega, \delta > 0,\ 0 < \varepsilon \ll 1.$$

*References*:


[1] J. Guckenheimer and Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer (1983) [Section 4.5]

[2] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, 2nd ed. Springer (2003) [Chapter 28]

# 10 Neutral saddle homoclinic bifurcation in three dimensions

Consider a generic 3D system depending on two parameters:

$$\dot{u} \; = \; F(u,\alpha) \;\; , \quad u \in \mathbb{R}^3, \, \alpha \in \mathbb{R}^2, \tag{10.1}$$

which has at $\alpha = 0$ an equilibrium $u = 0$ with eigenvalues

$$\nu \; < \; \mu \; < \; 0 \; < \; \lambda \;\; , \quad \sigma \; = \; \mu + \lambda \; = \; 0 \;\; .$$

and also has one orbit $\Gamma_0$ homoclinic to this equilibrium.

1. Discuss bifurcations in (10.1) happening for small $\|\alpha\|$. In particular, give arguments for existence of cyclic fold, period-doubling, and secondary homoclinic bifurcation curves emanating from the origin in the $\alpha$–plane, when $\Gamma_0$ is twisted.

2. Provide an example 3D ODE with this bifurcation.

*References*:


[1] S.N. Chow, B. Deng and B. Fiedler, Homoclinic bifurcation at resonant eigenvalues. *J. Dyn. Diff. Eq.* **2** 177–244 (1990)

[2] L.P. Shilnikov, A.L.Shilnikov, D.V. Turaev and L. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II. World Scientific (2001) [pp. 715–720 & 911–913]

## 11 Belyakov homoclinic bifurcation in three dimensions

Consider a generic 3D system depending on two parameters:

$$\dot{u} \;=\; F(u,\alpha)\;, \quad u \in \mathbb{R}^3,\, \alpha \in \mathbb{R}^2, \tag{11.1}$$

which has at $\alpha = 0$ an equilibrium $u = 0$ with eigenvalues

$$\nu \;=\; \mu \;<\; 0 \;<\; \lambda\;,$$

and also has one orbit $\Gamma_0$ homoclinic to this equilibrium.

1. Discuss bifurcations in (11.1) happening for small $\|\alpha\|$. In particular, give arguments for existence of infinitely-many cyclic fold, period-doubling, and secondary homoclinic bifurcation curves emanating from the origin in the $\alpha$–plane when $\sigma = \mu + \lambda > 0$.

2. Explain implications of such bifurcation points on the existence of complicated travelling waves in the FitzHugh–Nagumo model.

*References*:


[1] Yu.A. Kuznetsov, O. De Feo and S. Rinaldi, Belyakov homoclinic bifurcations in a tritrophic food chain model. *SIAM J. Appl. Math.* **62** 462–487 (2001) [Appendix A]

[2] Yu.A. Kuznetsov, *Applied Nonlinear Dynamics*. Utrecht University & University of Twente (2023) [Section 8.3, Exercise 2]

## 12 Lorenz system

1. Explain the origin of the Lorenz-63 ODEs:

$$\begin{cases} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= -bz + xy \end{cases} \tag{12.1}$$

   where $(\sigma, r, b)$ are positive parameters.

2. Analyse stability of equilibria in (12.1), in particular, derive a condition for a nonzero equilibrium to have a Hopf bifurcation.

3. Prove that the Hopf bifurcation in (12.1) is always subcritical.

4. Describe the sequence of global bifurcations of (12.1) leading to the Lorenz strange attractor and related 1D dynamics.

*References*:


[1] L.P. Shilnikov, A.L.Shilnikov, D.V. Turaev and L. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics*, Part II. World Scientific (2001) [pp. 877–880]

[2] J. Guckenheimer and Ph. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer (1983) [Sections 6.4 and 5.7]

[3] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Section 6.7 (Exercise E 6.7.3)] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect12.pdf`

[4] Yu.A. Kuznetsov, O. Diekmann, and W.-J. Beyn, *Dynamical Systems Essentials*. [On-line lecture notes, Section 7.6] `http://www.staff.science.uu.nl/~kouzn101/NLDV/Lect14.pdf`