Exercises 3. 10. 2012

1. A matrix $A \in M_{n\times n}(\mathbb{R})$ defines a linear differential equation $\dot{x} = Ax$, write L_A for the corresponding vector field. Next to the Lie bracket $[L_A, L_B]$ there is also the commutator $[A, B] = AB - BA$ of two matrices; show that

$$[L_A, L_B] \;=\; -L_{[A,B]} \;.$$

Furthermore show that

$$\varphi_t \circ \psi_s \;-\; \psi_s \circ \varphi_t \;=\; s\,t\,[A, B] \;+\; \mathcal{O}\left((s^2 + t^2)^{\frac{3}{2}}\right)$$

for the flows φ of L_A and ψ of L_B. Hence, $[L_A, L_B] = 0$ if these two flows commute.

2. Show that $\mathfrak{sl}_n(\mathbb{R}) = \{A \in M_{n\times n} \mid \operatorname{tr} A = 0\}$ is the Lie algebra of the Lie group $SL_n(\mathbb{R}) = \{S \in M_{n\times n} \mid \det S = 1\}$. To this end compute for a curve

$$\begin{array}{rccc} \gamma \;: & \mathbb{R} & \longrightarrow & M_{n\times n} \\ & t & \mapsto & \gamma(t) \end{array}$$

the corresponding tangent vector

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)\right|_{t=0}$$

and use $\gamma(t) = \exp tA$ to conclude that $\mathfrak{sl}_n(\mathbb{R}) = T_{\mathrm{Id}}SL_n(\mathbb{R})$. What are the consequences for the flow of a divergence-free vector field?

3. For $T \in GL_n(\mathbb{R})$ show that T has a real logarithm, i.e. $T = \exp A$ with $A \in M_{n\times n}(\mathbb{R})$, if and only if there exists $S \in GL_n(\mathbb{R})$ with $S^2 = T$. *Hint:* First note that also $T \in GL_n(\mathbb{C})$ and compute the complex logarithm. Assume (if necessary) that T is semi-simple.