Exercises 31. 10. 2012

1. Visualize the flow φ of

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \\ \dot{z} &= -z \end{aligned}$$

and show that φ_t maps the leaves

$$W^-(x,y) = \left\{ (\xi,\eta,\zeta) \in \mathbb{R}^3 \;\middle|\; \lim_{t\to\infty} \|\varphi_t(\xi,\eta,\zeta) - \varphi_t(x,y,0)\| = 0 \right\}$$

into each other.

2. Compute all center manifolds of

$$\begin{aligned} \dot{x} &= x^2 \\ \dot{y} &= -y \end{aligned}$$

and conclude that a center manifold is not unique. *Hint:* W^0 consist of the equilibrium, the positive x–axis and one additional trajectory.

3. Define on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ (a circle with radius $\frac{1}{2\pi}$) for a given rotation number $\rho \in \mathbb{R}$ the rigid rotation $R : x \mapsto x + \rho \pmod 1$. Show that for $\rho \notin \mathbb{Q}$ every orbit $\{x, R(x), R^2(x), \ldots\}$ lies dense in $\mathbb{T}$. What is true for $\rho \in \mathbb{Q}$?

   Let now $\dot{x}_1 = 1$, $\dot{x}_2 = \omega$ be a differential equation on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Show that for $\omega = \frac{p}{q} \in \mathbb{Q}$ (no common primes) all orbits are periodic, with (minimal) period q. Furthermore show that for $\omega \notin \mathbb{Q}$ every trajectory densely spins around the torus $\mathbb{T}^2$. *Hint:* Define a Poincaré mapping on $\{x_1 = 0\}$ and use the first part of this exercise. How can an orbit be represented within the square $[0,1]^2 \subseteq \mathbb{R}^2$?