Exercises 5. 12. 2012

1. On $\mathbb{T}^2$ consider the Poincaré section $C := \{x_1 = 0\}$ and a vector field X with flow φ_t for which the Poincaré mapping has the form $R : x_2 \mapsto x_2 + \rho \pmod 1$. In general the return time $\tau = \tau(x_2)$ shall not be constant. Construct a Poincaré section
$$D \;=\; \left\{ \varphi_{t(x_2)}(0, x_2) \;\middle|\; x_2 \in \mathbb{T} \right\}$$
for which the return time is constant. *Hint:* searching for a (periodic) translation $t = t(x_2)$ of time leads to $\varphi_\tau(D) = D$ with constant time $\tau \in \mathbb{R}$. Explicitly write this as an equation in t and τ and formulate conditions on ρ that yield a solution.

2. Let X be a vector field with multi-periodic torus V. Show that $X_{|V}$ is quasi-periodic if and only if V is the closure of any of its orbits.

3. Let
$$\begin{aligned} \dot{x} &= f(x,z) = \omega + \mathcal{O}(z) \\ \dot{z} &= g(x,z) = a(x) \cdot z + \mathcal{O}(z^2) \end{aligned}$$
be a vector field on $\mathbb{T}^n \times \mathbb{R}$ with $a(x) > 0$ for all $x \in \mathbb{T}^n$. The aim is to construct a co-ordinate transformation
$$\begin{aligned} \Phi : \mathbb{T}^n \times \mathbb{R} &\longrightarrow \mathbb{T}^n \times \mathbb{R} \\ (x,z) &\longmapsto (x, \lambda(x) \cdot z) \end{aligned}$$
that puts the vector field into Floquet form
$$\begin{aligned} \dot{x} &= \omega + \mathcal{O}(z) \\ \dot{z} &= \Omega z + \mathcal{O}(z^2) \end{aligned}$$
with $\Omega > 0$. To this end verify that this leads to the equation
$$\Omega \;=\; a + \langle \omega \mid \nabla_x \nu \rangle$$
in $\nu(x) := \ln \lambda(x)$ and give necessary conditions both for a formal solution and for a real-analytic solution.