Exercise 5 (Homothetic role of γ) By a scaling of the time t and of ω show that Theorem 3.1 only has to be proven for the case $\gamma = 1$. What happens to the bounded domain Γ as γ gets small?

Exercise 6 (Order of truncation) Following Exercise 5 we restrict to the case where $\gamma = 1$, taking the set Γ sufficiently large to contain a nontrivial 'Cantor set' of parameter values corresponding to $(\tau, 1)$-Diophantine frequencies. Maintaining the plus-notation consider the complexified domain

$$D = (\mathbb{T}^n + \tfrac{1}{2}\kappa + s) \times (\Gamma'_{\tau,1} + r),$$

see (3.10), assuming that $r = \frac{1}{2}s^{2\tau+2}$. For the order of truncation d take

$$d = \text{Entier}\left(s^{-2}\right).$$

(i) Show that for all integer vectors $k \in \mathbb{Z}^n$ with $0 < |k| < d$ one has $|k|^{\tau+1} \leq (2r)^{-1}$;

(ii) Next show that for all $\sigma \in \Gamma'_{\tau,1} + r$ and all k with $0 < |k| < d$ one has $|\langle \sigma, k\rangle| \geq \frac{1}{2}|k|^{-\tau}$;

(iii) As an example take $s_j = \left(\frac{1}{4}\right)^j, j \in \mathbb{Z}_+$, and express the order of truncation d_j as a function of j.

Exercise 7 (A normal form for families of circle maps) Given a 1-parameter family of circle maps $P_\lambda : \mathbb{T}^1 \to \mathbb{T}^1$ of the form

$$P_\lambda : x \mapsto x + 2\pi\beta + f(x, \lambda),$$

where x is counted mod 2π and where $f(x, 0) \equiv 0$. One has to show that by successive transformations of the form

$$H_\lambda : x \mapsto x + h(x, \lambda)$$

the x-dependence of P can be pushed away to higher and higher order in λ. For this appropriate conditions on β will be needed. Carry out the corresponding inductive process. What do you think the first step is? Then, concerning the Nth step, consider

$$P_{N,\lambda}(x) = x + 2\pi\beta + g(\lambda) + f_N(x, \lambda) + O(|\lambda|^{N+1}),$$

with $f_N(x, \lambda) = \tilde{f}(x)\lambda^N$ and look for a transformation $H = \text{Id} + h$, with $h(x, \lambda) = \tilde{h}(x, \lambda)\lambda^N$, such that in $H^{-1} \circ P_N \circ H$ the Nth order part in λ is x-independent. Formulate sufficient conditions on β, ensuring that the corresponding equation can be formally solved, in terms of Fourier series. Finally give conditions on β, such that in the real analytic case we obtain real analytic solutions h. Explain your arguments.