The study of planar dynamical systems at infinity

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1 Introduction

Phase plane analysis often starts by locating equilibria and studying their local and global behaviour, but for polynomial systems this often is not enough. Orbits might shoot to infinity and the geometry at infinity has the power to dictate the global dynamics. The Poincaré compactification produces a systematic way to present infinity by mapping $\mathbb{R}^2$ onto the unit sphere, in turn mapping infinity to its equator. When doing this, one changes questions about unbounded orbits into ones about the flow near the equator. Here, so-called infinite singular points can exist, which are the 'infinite' equivalent to ordinary equilibria. These singularities can be classified much like ordinary equilibria, but degeneracies often require techniques beyond linearisation. In this essay, we outline the compactification procedure, derive the equations used to study the flow near infinity and discuss how infinite singular points can put constraints on global phenomena such as trapping regions and limit cycles.

2 The Poincaré compactification of the plane

We are interested in studying planar systems at infinity. To draw what happens at infinity, we need to compactify the plane. Note that we will only work with polynomial planar systems. I.e. for the planar system

$$\begin{cases} \dot{x} &= P(x,y) \\ \dot{y} &= Q(x,y) \end{cases} \tag{A}$$

we have that P, Q are both polynomials. We denote $X(x,y) = (P(x,y), Q(x,y))$ as the polynomial vector field corresponding to this planar system. The degree of X is the maximal degree of P and Q, i.e.

$$\deg X = \max\{\deg P, \deg Q\}.$$

In the rest of this article, we will denote our plane with (x_1, x_2) instead of the usual (x, y).

For the compactification, consider the plane $\mathbb{R}^2$ as the plane defined by $\{y \in \mathbb{R}^3 | (y_1, y_2, y_3) = (x_1, x_2, 1)\} \subset \mathbb{R}^3$. This is just the (x_1, x_2)-plane shifted to $y_3 = 1$. We then look at the unit sphere, usually called the *Poincaré sphere*. It is defined as $\mathbb{S}^2 = \{y \in \mathbb{R}^3 | y_1^2 + y_2^2 + y_3^2 = 1\}$. We can divide this into three sections, namely the upper and lower hemispheres and the equator:

$$\begin{aligned} H_+ &= \{y \in \mathbb{S}^2 | y_3 > 0\} \\ H_- &= \{y \in \mathbb{S}^2 | y_3 < 0\} \\ \mathbb{S}^1 &= \{y \in \mathbb{S}^2 | y_3 = 0\}. \end{aligned}$$

With these definitions, we can start projecting onto the Poincaré sphere. We do this for the upper and lower hemiphere seperately.

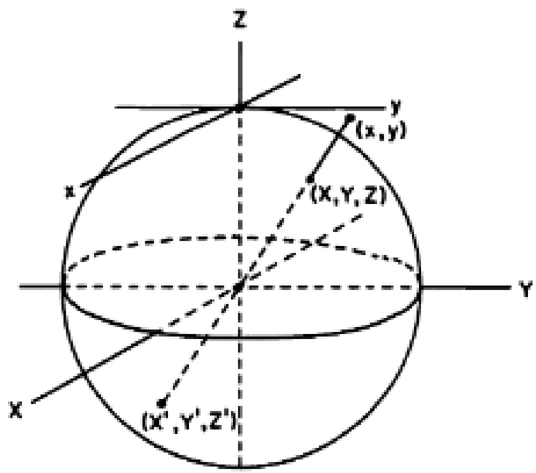

Figure 1: A visualisation of the Poincaré compactification onto the Poincaré sphere. The point $(x, y, z) = (y_1, y_2, y_3) \in H_+$ lies on the upper hemisphere, while $(x', y', z') = (-y_1, -y_2, -y_3) \in H_-$ lies on the lower hemisphere.[2]

For the upper hemisphere, we define a function $f^+ : \mathbb{R}^2 \to \mathbb{S}^2$ as

$$f^+(x_1, x_2) = \Big(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}\Big)$$

where $\Delta(x) = \sqrt{x_1^2 + x_2^2 + 1}$. Similarly, we define $f^-(x) : \mathbb{R}^2 \to \mathbb{S}^2$ as

$$f^-(x_1, x_2) = \Big(\frac{-x_1}{\Delta(x)}, \frac{-x_2}{\Delta(x)}, \frac{-1}{\Delta(x)}\Big).$$

The idea of these projections is to draw a line through the centre of $\mathbb{S}^2$, i.e. just $(0, 0, 0)$, and a point on $\mathbb{R}^2$. Wherever this line intersects either the upper or lower hemisphere is where we map the point of the plane to with f^+ and f^- respectively. This is scetched in figure 1.

We note both of these functions are diffeomorphisms, so the induced flow on $\mathbb{S}^2$ is topologically conjugate to that of our original plane $\mathbb{R}^2$. We can explicitly give this flow by using the chain rule. On H_+, we define $y(t) = f^+(x(t))$. With this, we find

$$\dot{y} = Df^+(x(t))\dot{x}(t) = Df^+(x)X(x). \tag{1}$$

I.e. the induced vector field (or dynamical system, whichever you prefer) on H_+ is given by

$$\overline{X}(y) \equiv Df^+(x)X(x)$$

with $y = f^+(x)$. Similarly, we can define the induced vector field on H_- as

$$\overline{X}(y) \equiv Df^-(x)X(x).$$

In short, we have that $\overline{X}$ is a vector field on $\mathbb{S}^2 \setminus \mathbb{S}^1$.

We call d the maximal degree of P and Q, so by definition $\deg X = d$. We note that as we approach the equator after our projection ($y_3 \to 0$), we have that $\Delta(x) \to \infty$, i.e. $\|x\| = \sqrt{x_1^2 + x_2^2} \to \infty$. This would imply that adding the equator would correspond with adding points at infinity, which is indeed true. One major problem with this is that the induced vector field does not necessarily stay bounded when $\|x\| \to \infty$, as X grows as $\|x\|^d$, while f^+ grows as $\frac{1}{\|x\|}$ for large x. This together results in the induced vector field growing as $\|x\|^{d-1}$ for large x, clearly not staying bounded. One way to overcome this problem is to reparametrise time. In particular, by multiplying our induced vector field by a factor of $\rho(x) = x_3^{d-1}$ where x_3 is the height on the sphere (so $x_3 = y_3 = f_3^+(x) = 1/\Delta(x)$). In other words, this factor $\rho(x)$ grows as $\frac{1}{\|x\|^{d-1}}$ for large x. With this, we can rescale the induced vector field to find $\rho\overline{X} \sim (1/\|x\|^{d-1})\,\|x\|^{d-1} \sim O(1)$. I.e. when reparametrising time, we can make sure the vector field stays finite.

Making sure the flow near and on the equator is finite makes sure we can extend our induced vector field continuously, which gives the oppurtunity to view 'infinity' as a legitimate point in the dynamical system. One last remark is that because of the time reparametrisation, we have lost the topological conjugacy between the two systems. We still have a topological equivalence, which will suffice for making phase portraits, but caution is needed when calculating properties like eigenvalues of equilibria and periods of cycles.

We will now focus on the extension of $\mathbb{S}^2 \setminus \mathbb{S}^1$ to the full sphere, by adding the points at infinity. For this, we will use charts. Simply stated, charts are tuples (U, φ) of open patches U and smooth maps φ from said open patches to open sets of $\mathbb{R}^n$, with a smooth inverse on their image. In our case, these charts map open patches of the unit sphere to the plane. One can view this as a local 'flattening' of parts of the sphere, making it possible to describe points in the chart with two real coordinates.

For $\mathbb{S}^2$, we use six local charts, namely $U_k = \{y \in \mathbb{S}^2 | y_k > 0\}$, $V_k = \{y \in \mathbb{S}^2 | y_k < 0\}$ for $k = 1, 2, 3$. I.e. one can split $\mathbb{S}^2$ through each of the axes, resulting in U_k and V_k. These charts have corresponding maps $\varphi_k(y) = -\psi_k(y) = (\frac{y_m}{y_k}.\frac{y_n}{y_k}) = (u, v)$ for $m < n$ and $m, n \neq k$ for U_k and V_k respectively. These maps have the same idea as our initial compactification to the Poincaré sphere, namely a central projection. For U_1 for example, we draw a line from the centre of $\mathbb{S}^2$ to a point on U_1. Where this line intersects the hyperplane $y_1 = 1$ is where we send that point of U_1. With this idea, one can view the (u, v) coordinates as corresponding with the direction and distance of the new point. We can visualise this by composing the corresponding map with f^+:

$$\begin{aligned}(u, v) &= \varphi_1(f^+(x)) \\ &= \varphi_1\Big(\frac{x_1}{\Delta(x)}, \frac{x_2}{\Delta(x)}, \frac{1}{\Delta(x)}\Big) \\ &= \Big(\frac{x_2/\Delta(x)}{x_1/\Delta(x)}, \frac{1/\Delta(x)}{x_1/\Delta(x)}\Big) \\ &= \Big(\frac{x_2}{x_1}, \frac{1}{x_1}\Big).\end{aligned} \tag{2}$$

Note x_2/x_1 can be considered the slope or direction of the line, while $1/x_1$ is inversely proportional to the distance ($v \to 0 \implies |x_1| \to \infty$).

More concretely, when we have $X(x) = (P(x_1, x_2), Q(x_1, x_2))$ and $\overline{X}(y) = Df^+(x)X(x)$ with $y = f^+(x)$, we calculate

$$D\varphi_1(y)\overline{X}(y) = D\varphi_1(y) \circ Df^+(x)X(x) = D(\varphi_1 \circ f^+)(x)X(x).$$

We will denote this expression as $\overline{X}|_{U_1}$. Calculating this explicitly, we find

$$\begin{aligned}\overline{X}|_{U_1} &= D(\varphi_1 \circ f^+)(x)X(x) \\ &= \begin{pmatrix} -\frac{x_2}{x_1^2} & \frac{1}{x_1} \\ -\frac{1}{x_1^2} & 0 \end{pmatrix} \begin{pmatrix} P(x_1, x_2) \\ Q(x_1, x_2) \end{pmatrix} \\ &= \frac{1}{x_1^2} \begin{pmatrix} -x_2 P(x_1, x_2) + x_1 Q(x_1, x_2) \\ -P(x_1, x_2) \end{pmatrix}.\end{aligned}$$

We can now substitute the expressions $x_1 = 1/v$ and $x_2 = u/v$ to get

$$\overline{X}|_{U_1} = v^2 \begin{pmatrix} -\frac{u}{v}P(\frac{1}{v}, \frac{u}{v}) + \frac{1}{v}Q(\frac{1}{v}, \frac{u}{v}) \\ -P(\frac{1}{v}, \frac{u}{v}) \end{pmatrix}.$$

Or otherwise stated, our compactified dynamical system on chart U_1 is given by

$$\begin{cases} \dot{u} &= v^2[-\frac{u}{v}P(\frac{1}{v}, \frac{u}{v}) + \frac{1}{v}Q(\frac{1}{v}, \frac{u}{v})] \\ \dot{v} &= -v^2 P(\frac{1}{v}, \frac{u}{v}). \end{cases}$$

We again run into the same problem as before: this vector field is not necessarily bounded. In particular, for $v \to 0$ this expression might blow up. We can fix this with the same multiplication by $\rho(y)$, where

$$\begin{aligned}\rho(y) = y_3^{d-1} &= \frac{1}{\Delta(x)^{d-1}} \\ &= \frac{1}{\left(\sqrt{\frac{1+u^2+v^2}{v^2}}\right)^{d-1}} \\ &= \frac{v^{d-1}}{(\sqrt{1+u^2+v^2})^{d-1}}.\end{aligned}$$

Writing out the multiplication, we get

$$\rho\overline{X}|_{U_1} = \frac{v^{d+1}}{(\sqrt{1+u^2+v^2})^{d-1}} \begin{pmatrix} -\frac{u}{v}P(\frac{1}{v},\frac{u}{v}) + \frac{1}{v}Q(\frac{1}{v},\frac{u}{v}) \\ -P(\frac{1}{v},\frac{u}{v}) \end{pmatrix}.$$

Note this multiplication makes sure every possible v in the denominator is cancelled by v^{d+1}, as the maximal degree of P and Q is d.

To simplify things further, we would like to get rid of the root in the denominator. This is possible with another time change. To ease notation, we will denote $1/(\sqrt{1+u^2+v^2})^{d-1} = \alpha(u,v)$. We define the new time as

$$d\tau = \alpha(u(t),v(t))dt.$$

rewriting the dynamical system in this new time results in

$$\begin{cases} \dot{u} &= \frac{du}{d\tau} = v^{d+1}[-\frac{u}{v}P(\frac{1}{v},\frac{u}{v}) + \frac{1}{v}Q(\frac{1}{v},\frac{u}{v})] \\ \dot{v} &= \frac{dv}{d\tau} = -v^{d+1}P(\frac{1}{v},\frac{u}{v}). \end{cases}$$

We note this same calculation can be done for all charts seperately, where f^- should be used for (V_k,ψ_k). As we reparametrised time twice, we again are left with a topological equivalence, which will suffice for us.

Writing out the expressions for charts (U_k,φ_k), we have

$$\begin{cases} \dot{u} &= v^d[-uP(\frac{1}{v},\frac{u}{v}) + Q(\frac{1}{v},\frac{u}{v})] \\ \dot{v} &= -v^{d+1}P(\frac{1}{v},\frac{u}{v}) \end{cases} \tag{B}$$

for (U_1,φ_1),

$$\begin{cases} \dot{u} &= v^d[P(\frac{u}{v},\frac{1}{v}) - uQ(\frac{u}{v},\frac{1}{v})] \\ \dot{v} &= -v^{d+1}Q(\frac{u}{v},\frac{1}{v}) \end{cases} \tag{C}$$

for (U_2,φ_2) and

$$\begin{cases} \dot{u} &= P(u,v) \\ \dot{v} &= Q(u,v) \end{cases} \tag{D}$$

for (U_3,φ_3). Note the expressions for (V_k,ψ_k) are the same as the ones above, multiplied by a factor $(-1)^{d-1}$. This shows the behaviour on the upper and lower hemisphere are practically the same, with at most a change in direction of orbits (when d is even). Because of this, it suffices to study only $H_+ \cup \mathbb{S}^1$. This object is often called the *Poincaré disc*. Thus, all calculations can be done in the charts (U_k,φ_k) for $k = 1,2,3$.

3 Infinite singular points

We will now look at different types of equilibria which lie on $\mathbb{S}^1$, which are often called infinite singular points. We remark that if y is an infinite singular point, then so is $-y$. This comes from the symmetry through the origin of our compactification. Remember that opposite points lie on opposite charts (U_k and V_k), where local behaviour only differs by a factor of $(-1)^{d-1}$. So in particular, the stability of y and $-y$ are equal when d is odd, while they differ when d is even. Also remember the vector field on the top and bottom hemisphere of $\mathbb{S}^2$ is symmetric through the origin, thus it is sufficient to study flows in the system on only one of the hemispheres.

We look at an infinite singular point $(u, 0) \in \mathbb{S}^1$. We can decompose $X(x)$ into homogeneous polynomials P_i, Q_i of degree $i = 0, 1, \ldots, d$. So, $P = P_0 + P_1 + \cdots + P_d$ and $Q = Q_0 + Q_1 + \cdots + Q_d$. For a singular point, whether at infinity or not, we need $\dot{u} = \dot{v} = 0$. We look at $\mathbb{S}^1 \cap U_1$ specifically. Recalling system (B), we can immediately deduce $\dot{v} = 0$, as it is a multiplication of v^{d+1} with something of order d. So, we can simplify our requirements, saying $\dot{u} = v^d[-uP(\frac{1}{v}, \frac{u}{v}) + Q(\frac{1}{v}, \frac{u}{v})] = 0$. Decomposing P and Q into their homogeneous polynomials, we find

$$\dot{u} = \sum_{i=0}^{d} v^d[-uP_i(\frac{1}{v}, \frac{u}{v}) + Q_i(\frac{1}{v}, \frac{u}{v})]$$

where we deduce all terms except P_d, Q_d vanish for the same reasons as before. Thus, this all simplifies to

$$\dot{u} = v^d[-uP_d(\frac{1}{v}, \frac{u}{v}) + Q_d(\frac{1}{v}, \frac{u}{v})] = 0.$$

We note we can extract v^{-d} out of P_d and Q_d, resulting in

$$F(u) \equiv Q_d(1, u) - uP_d(1, u) = 0.$$

Note that on $\mathbb{S}^1 \cap V_1$ the requirement is the same, as the only difference is the direction of time, which is unimportant for this equation. Also note this statement works both ways: if $F(u) = 0$, then $(u, 0)$ is an infinite singular point.

We can follow the same logic for $\mathbb{S}^1 \cap U_2$ and $\mathbb{S}^2 \cap V_2$, resulting in $(u, 0)$ being an infinite singular point iff

$$G(u) \equiv P_d(u, 1) - uQ_d(u, 1) = 0.$$

Using these expressions, we can calculate the jacobians in both cases:

$$J = \begin{pmatrix} F'(u) & Q_{d-1}(1, u) - uP_{d-1}(1, u) \\ 0 & -P_d(1, u) \end{pmatrix} \qquad \text{(in } \mathbb{S}^1 \cap (U_1 \cup V_1)\text{)} \tag{3}$$

$$J = \begin{pmatrix} G'(u) & P_{d-1}(u, 1) - uQ_{d-1}(u, 1) \\ 0 & -Q_d(u, 1) \end{pmatrix} \qquad \text{(in } \mathbb{S}^1 \cap (U_2 \cup V_2)\text{).} \tag{4}$$

Note as both jacobians are upper triangular, there cannot exist a focus or centre infinite singular point. Points that can exist, can be seperated into three categories:

1. Hyperbolic infinite singular points. This is when both eigenvalues have a nonzero real part (i.e. are nonzero in this case). Note this implies $\lambda_1 = F'(u), \lambda_2 = -P_d(1, u)$ are both nonzero in the first case and $\lambda_1 = G'(u), \lambda_2 = -Q_d(u, 1)$ are both nonzero in the second case. Note that if one of these infinite singular points is a saddle, then the equator $\{(u, v) \in \mathbb{S}^2 | v = 0\}$ is a stable or unstable manifold. This results from the fact that the equator is invariant (whenever $v = 0$, we have $\dot{v} = 0$) and tangent to the eigendirection $(1, 0)$. The stability of this manifold depends on the sign of the eigenvalues of the jacobian.

2. Semi-hyperbolic infinite singular points. This is the case when one of the eigenvalues is zero and the other nonzero. In general, these cases result in saddle-node type equilibria at infinity. There can exist two of these, depending on which eigenvalue is zero. This is shown in figure 2. Note that for semi-hyperbolic infinite singular points the equator is also a stable or unstable manifold, depending on the sign of the eigenvalues.

3. Nilpotent infinite singular points. This is when both eigenvalues are zero, requiring more in depth calculations to determine what is happening. One thing we do know immediately is that nilpotent versions of foci and centres are still not possible for the same reasons as before. The most used technique for studying nilpotent infinite singular points is called 'Blow-up', which we will explain in part 4.

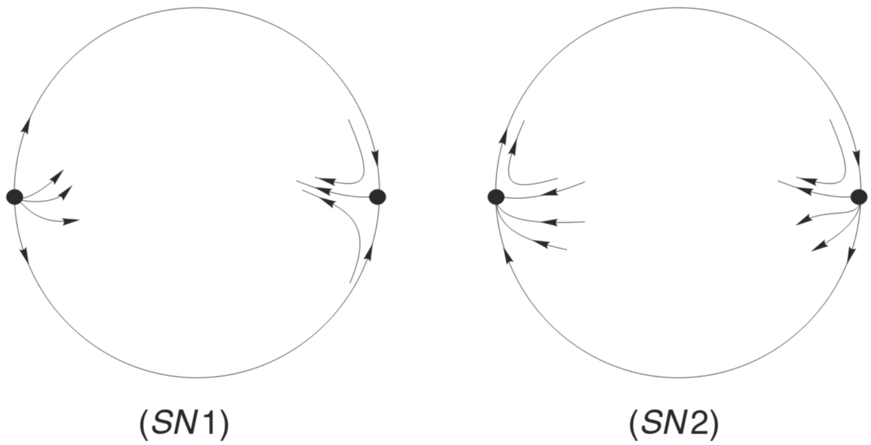

Figure 2: Two types of saddle-node infinite singular points. SN1 corresponds with the case that $\lambda_1 = F'(u) = 0$ (or $\lambda_1 = G'(u) = 0$), while SN2 corresponds with the case where $\lambda_2 = 0$.[1]

Note all infinite singular points of category 1 and 2 which can exists are easily calculated, just like how it is done in the finite case.

4 Blow-up techniques

As stated before, nilpotent infinite singular points can be quite difficult. One way to simplify them are the so-called blow-up techniques. We will talk about two of them: polar and quasihomogeneous. We start with the polar blow-up technique, and we will explain it in a finite setting.

Suppose there exists a degenerate equilibrium (nilpotent or with zero jacobian) at $(0,0) \in \mathbb{R}^2$. Instead of trying to determine what is happening at that point, we replace it by a small circle such that we can study every direction seperately. Formally, we replace a small neighbourhood of $(0,0)$ with a circle $(r,\theta) \in [0,\varepsilon) \times [0,2\pi)$ with ε very small. We do this by reparamtetrising that neigbourhood with $(x_1,x_2) = (r\cos\theta, r\sin\theta)$. Note that $r = 0$ corresponds with our blown-up point. After this substitution, you are often left with a factor of r^k for some $k \in \mathbb{Z}$. By reparametrising time ($d\tau = r^k dt$) we can remove this, resulting in a well-defined vector field at $r = 0$. Written more explicitly, we say we have the following planar system:

$$\begin{cases} \dot{x}_1 &= f(x_1,x_2) \\ \dot{x}_2 &= g(x_1,x_2). \end{cases}$$

As we work with polynomial vector fields, we can decompose f and g into homogeneous parts, where k is the lowest nonzero degree. We can then substitute our polar coordinates using the known formulas

$$\dot{r} = f\cos\theta + g\sin\theta, \qquad \dot{\theta} = \frac{-f\sin\theta + g\cos\theta}{r}. \tag{5}$$

Note that the lowest degree term in these expressions are r^k for $\dot{r}$ and r^{k-1} for $\dot{\theta}$. We now reparametrise time with $d\tau = r^{k-1}dt$, resulting in

$$\frac{dr}{d\tau} = \frac{\dot{r}}{r^{k-1}} \sim r, \qquad \frac{d\theta}{d\tau} = \frac{\dot{\theta}}{r^{k-1}} \sim \text{'constant in } r\text{'} \tag{6}$$

with other non-important higher order terms.

After this, we can look at this constant term of $d\theta/d\tau$ (i.e. when $r = 0$) and determine its zeros. These correspond with directions in which orbits can approach or leave the equilibrium. In between these zeros we can talk about sectors, namely hyperbolic (saddle), parabolic (node) and elliptic (centre) sectors. Again, in our cases on infinity, we will not encounter the third kind. We can then view these sections radially, i.e. with our $dr/d\tau$ expression. With this, we can determine the behaviour in each section. This ultimately gives us the needed information to draw phase portraits. We can use almost exactly this theory and apply it to infinite singular points. We just use the blow-up techniques on our chart coordinates.

A more complex version of this polar blow-up technique is called the quasihomogeneous blow-up or the Poincaré-Lyapunov compactification. The idea is the same as in the polar case, with the only exception of added 'weights' to directions. Simply stated, instead of just parametrising with $x_1 = r\cos\theta, x_2 = r\sin\theta$, one can add weights to the x and y direction with the parametrisation

$$x_1 = r^\alpha \cos\theta, \qquad x_2 = r^\beta \sin\theta$$

with $\alpha, \beta \in \mathbb{N}$. When chosen correctly, the right α and β values can make sure that degenerate infinite singular points can simplify into elementary ones, or even disappear completely. This is however mostly a guessing game, with no theorem telling you which values are advantegeous. Usually one looks at the scaling of the lowest order terms. For example: for a system where the lowest order terms are r and r^2 for x and y respectively, the weighted blow-up will most likely be with $\alpha = 1$ and $\beta = 2$. When studying nilpotent infinite singular points, quasihomogeneous is usually the second step, right after the polar blow-up.

To conclude this essay, we will give an example of how the theory of infinite singular points and blow-up techniques can be used. We will look at the following planar system:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + hx_2 - hx_2^3 \end{cases}$$

with $h > 0$. We will prove this system has a stable unit cycle. First, we look at the usual interesting finite information. We observe the only equilibrium in this planar system is $(0, 0)$ with jacobian

$$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & h - 3hx_2^2 \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & h. \end{pmatrix}$$

Calculating its eigenvalues, we find $\lambda = \frac{h}{2} \pm \frac{1}{2}\sqrt{h^2 - 4}$. This results in being an unstable focus for $0 < h < 2$ and an unstable node for $h \geq 2$. Thus, for all $h > 0$, this equilibrium is unstable. We will now look at the infinite case. Remember that $P(x_1, x_2) = x_2$ and $Q(x_1, x_2) = -x_1 + hx_2 - hx_2^3$. We can skip the abstraction and start calculating the behaviour on every interersting chart. Note beforehand that $d = \deg X(x) = 3$, which implies the behaviour on U_k is equal to that on V_k for $k = 1, 2, 3$. We start by looking at chart U_2, i.e. the chart in the where $y_2 > 0$. Looking at system (C) and plugging in $x_1 = u/v$ and $x_2 = 1/v$, we find

$$\begin{cases} \dot{u} &= -h + (1 - h + u^2)v^2 \\ \dot{v} &= (u - h)v^3 + hv. \end{cases}$$

We are interested when $x_2 \to \infty$, or otherwise stated, when $v \to 0$. Note that for small v, $\dot{v} \approx hv > 0$. Simply said, for small v, we find that the equator ('infinity') is repelling. To formalise this statement we also look at the jacobian (4). Plugging in our numbers, we get

$$J = \begin{pmatrix} h & * \\ 0 & h \end{pmatrix}.$$

I.e. we find that the eigenvalues of the infinite singular point(s) in chart U_2 all have eigenvalues $\lambda_1 = \lambda_2 = h > 0$. Hence they are repelling.

We now look at chart U_1. Now taking system (B) and substituting $x = 1/v$ and $y = u/v$, we find

$$\begin{cases} \dot{u} &= -hu^3 + (hu - u^2 - 1)v^2 \\ \dot{v} &= -uv^3. \end{cases}$$

To analyse what happens at and around $v = 0$, we now have to use a blow-up technique. In this case, a polar blow-up suffices. We make the substitution $u = r\cos\theta$ and $v = r\sin\theta$. As we are interested in chart U_1 where $y_1 > 0$, we have that $\theta \in [0, \pi]$ such that $v \geq 0$. We get

$$\begin{cases} \dot{u} &= -r^2\sin^2\theta + O(r^3) \\ \dot{v} &= -r^4\cos\theta\sin^3\theta \end{cases}.$$

Substituting these expressions into (5), we find

$$\begin{cases} \dot{r} &= -r^2\sin^2\theta\cos\theta + O(r^3) \\ \dot{\theta} &= r\sin^3\theta + O(r^2). \end{cases}$$

We now make the time change explained in (6), where $k = 2$ was the lowest nonzero degree. We find

$$\begin{cases} r' &= -r\sin^2\theta\cos\theta + O(r^2) \\ \theta' &= \sin^3\theta + O(r) \end{cases}$$

where r', θ' are the new time derivatives. We can now set $r = 0$ and look at the behaviour of θ'. Observe that $\sin^3\theta = 0$ if and only if $\theta = 0, \pi$. So, the whole interior of our starting interval $[0, \pi]$ is one section. Note in particular that $\theta' = \sin^3\theta > 0$ in this section, which implies that every orbit in this section eventually moves to $\theta = \pi$.

We then look at the radial aspect when r is very small, so $r' \approx -r\sin^2\theta\cos\theta$. Note that $r > 0, \sin^2\theta > 0$ in the whole section and $\cos\theta > 0$ for $\theta \in [0, \pi/2)$ and $\cos\theta < 0$ for $\theta \in (\pi/2, \pi]$. This implies $r' < 0$ for $\theta \in [0, \pi/2)$ and $r' > 0$ for $\theta \in (\pi/2, \pi]$. Simply stated, for the smaller half of the θ-interval, 'infinity' attracts, while it repells on the second half. But remember $\theta' > 0$ on the whole interval, so every orbit starting in this section will eventually end up in the second half of the interval, resulting in a repelling force of the equator of $\mathbb{S}^2$.

Note that as $d = 3$ is odd, we have that the behaviour on V_1 and V_2 is equal to that on U_1 and U_2. Thus, the equator repells all orbits close to it. We can now use this to prove there exists a stable limit cycle. This is done by a simple Poincaré-Bendixon argument. We can make an (arbitrarily) small closed curve around $(0, 0)$ such that there still exists an outward flow everywhere. In the same way we can make an arbitrarily large closed curve close to infinity such that it has an inward flow everywhere. With these, we can make a trapping annulus. As there exists no other equilibria in this annulus, there must exist a stable limit cycle inside, concluding this example.

5 Conclusion

In this essay we showed how the Poincaré compactification could be used to change questions about unbounded orbits into local analysis on the equator of the unit sphere. Infinite singular points arise as the 'infinite' equivalent to equilibria, which can be classified using chart coordinates. When linearization fails, blow-up techniques are used to resolve degeneracies by seperating directions, enabling the study each direction seperately. In short, the behaviour around infinity becomes accesible with these techniques, providing information about the global dynamics. For example by enabling the construction of trapping regions that imply stable limit cycles.

References

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