Quasi-periodic bifurcations in reversible systems

Dedicated to Henk Broer at the occasion of his 60th birthday

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Abstract

Invariant tori of integrable dynamical systems occur both in the dissipative and in the conservative context, but only in the latter the tori are parametrised by phase space variables. This allows for quasi-periodic bifurcations within a single given system, induced by changes of the normal behaviour of the tori. It turns out that in a non-degenerate reversible system all semi-local bifurcations of co-dimension 1 persist, under small non-integrable perturbations, on large Cantor sets.

1 Introduction

A dynamical system is reversible if it admits an involutive symmetry G that maps orbits $z(t)$ to $(Gz)(t) = z(-t)$. Typical examples of reversible systems are second order equations, where one cannot infer from the dynamics if time is going backwards, see [21, 25, 19] and references therein. This includes many Hamiltonian systems, in particular simple mechanical systems where the Hamiltonian is the sum of the (quadratic) kinetic energy and a positional potential energy. However, it turned out that reversible systems share many properties of Hamiltonian systems even if they are not Hamiltonian themselves.

An important aspect are periodic orbits forming continuous families, and more generally Kolmogorov–Arnol'd–Moser (KAM)-theory. On an invariant torus with dense quasi-periodic orbits the reversor G has to act as $-\mathrm{id}$ and such tori are parametrised by submanifolds on which G acts as the identity. In the sequel we assume that suitable co-ordinates $(x, y) \in \mathbb{T}^n \times \mathbb{R}^m = (\mathbb{R}^n/\mathbb{Z}^n) \times \mathbb{R}^m$ around an invariant n–torus $\{y = y_0\}$ have been chosen and work with the reversor $G(x, y) = (-x, y)$. Then reversibility implies for

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{1}$$

that f is even in x and g is odd in x, in particular $g(0, y) = 0$ for all $y \in \mathbb{R}^m$. We call the vector field (1) integrable if it is equivariant with respect to the translational torus action

$$(x, y) \quad \mapsto \quad (x + \xi, y) \enspace ,$$

whence $g(x, y) \equiv g(y) \equiv 0$ vanishes identically and the phase space is fibrated into invariant tori $\mathbb{T}^n \times \{y\}$. The flow on such a torus is parallel (or conditionally periodic) with frequency vector $\omega(y) := f(y) \equiv f(x, y)$ and in particular quasi-periodic if $\omega(y)$ is rationally independent, every orbit being dense in $\mathbb{T}^n \times \{y\}$. In this setting KAM-theory provides the following result.

Theorem 1 [20, 21, 26, 10] *If the frequency mapping $\omega : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is non-degenerate, then most tori survive a small reversible perturbation.*

The classical situation is $m = n$ where one can formulate non-degeneracy

$$\det D\omega(y) \neq 0 \tag{2}$$

in the sense of Kolmogorov. Then ω is a local diffeomorphism and one can pull back the whole geometry of the frequency space into the phase space. Resonant tori, with frequency vector ω satisfying

$$\langle k \mid \omega \rangle := k_1\omega_1 + \ldots + k_n\omega_n = 0$$

for a non-zero $k \in \mathbb{Z}^n$, should not be considered as dynamical objects as they are merely disjoint unions of lower-dimensional tori that do have a dense orbit. Correspondingly, persistence under arbitrary perturbations can only be proven if ω satisfies a strong form of non-resonance. In the sequel we work with Diophantine conditions

$$\bigwedge_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} |2\pi\langle k \mid \omega\rangle| \geq \frac{\gamma}{|k|^\tau} \tag{3}$$

where $\tau > n - 1$ and $\gamma > 0$. For the weaker Bruno conditions see [31, 24, 16, 22]. While for bounded open $\Sigma \subseteq \mathbb{R}^n$ the set $\Sigma_{\gamma,\tau}$ of Diophantine frequency vectors is topologically small, its relative measure tends to 1 as $\gamma \to 0$. This remains unchanged when pulling back $\Sigma_{\gamma,\tau}$ with the local diffeomorphism ω into phase space. The same conclusion remains true if $m \geq n + 1$ and the frequency mapping $\omega : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a submersion.

It is instructive to study the geometric properties of the set $\Sigma_{\gamma,\tau}$ defined by (3). If $\omega \in \Sigma_{\gamma,\tau}$ then also $s\omega \in \Sigma_{\gamma,\tau}$ for all $s \geq 1$. This allows to relax the non-degeneracy condition (2), instead of all partial derivatives of ω spanning $\mathbb{R}^n$ it suffices that

$$< \omega, \frac{\partial \omega}{\partial y_1}, \ldots, \frac{\partial \omega}{\partial y_m} > = \mathbb{R}^n \tag{4}$$

which in particular allows for $m = n - 1$. The surviving tori no longer retain their frequency vector $\omega(y)$ but still have the same frequency ratios

$$[\omega(y)] := [\omega_1(y) : \omega_2(y) : \ldots : \omega_n(y)] ,$$

see [20, 8, 10]. The resulting Cantor dust parametrising the persistent invariant tori again has relative measure

$$\frac{\text{meas}([\omega]^{-1}\Sigma_{\gamma,\tau})}{\text{meas}([\omega]^{-1}\Sigma)} \overset{\gamma \to 0}{\longrightarrow} 1$$

tending to full measure.

The Diophantine conditions (3) exclude small neighbourhoods of (a dense set of) hyperplanes. What has to be avoided is that a large portion of the image $\omega(\mathbb{R}^m)$ of the frequency mapping vanishes in these resonance gaps. The conditions (2) and (4) enforce this by requiring $\omega(\mathbb{R}^m)$ to be transverse to $\Sigma_{\gamma,\tau}$, but the (linear) geometry of the latter already ensures this if $\omega(\mathbb{R}^m)$ is sufficiently curved. This allows to further relax to the non-degeneracy condition

$$< \frac{\partial^{|\ell|}\omega}{\partial y^\ell} \;\Big|\; \ell \in \mathbb{N}_0^m > \;=\; \mathbb{R}^n \tag{5}$$

of Rüssmann, cf. [23, 10, 24]. In the perturbed system one finds invariant tori with frequency vectors close to the unperturbed ones, though one should no longer speak of surviving tori. Choosing $\tau > nL - 1$ in (3), where L is the highest dervative needed in (5), ensures that the relative measure of invariant tori still tends to 1 as the size of the perturbation goes to 0.

The next section shows how these results can be generalized to non-degenerate lower-dimensional tori. The normal linear behaviour of $\mathbb{T}^n \times \{y\} \times \{0\} \subseteq \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ is elliptic or hyperbolic (or possibly a superposition of these in case $p \geq 2$) and bifurcations occur at tori with multiple eigenvalues on the imaginary axis. In section 3 the frequency-halving and quasi-periodic pitchfork bifurcations are studied, here the tori themselves persist and only the normal linear behaviour changes. During the quasi-periodic centre-saddle bifurcation of section 4 the elliptic and hyperbolic tori that meet at parabolic tori cease to exist. In the concluding section 5 these bifurcations are put into context. As an example the gaps left open by theorem 1 are addressed, where quasi-periodic bifurcations allow to shed some light on the dynamics.

2 Lower-dimensional tori

To allow for a non-trivial normal behaviour of the tori we extend (1) to

$$\begin{aligned} \dot{x} &= f(x,y,z) \\ \dot{y} &= g(x,y,z) \\ \dot{z} &= h(x,y,z) \end{aligned} \tag{6}$$

on $\mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ with reversing symmetry

$$G(x,y,z) \;=\; (-x, y, Rz) \;\; .$$

Here R is a linear involution on $\mathbb{R}^{2p}$ with $\dim \operatorname{Fix}(R) = p$, whence

$$\mathbb{R}^{2p} \;=\; \operatorname{Fix}(R) \,\oplus\, \operatorname{Fix}(-R) \;\; .$$

For the moment we refrain from changing co-ordinates to adapt to this splitting. In the unperturbed integrable case the right hand side of (6) is x–independent and reversibility yields

$$
\begin{aligned}
f(y, Rz) &= f(y, z) \\
g(y, Rz) &= -g(y, z) \\
h(y, Rz) &= -Rh(y, z)
\end{aligned}
$$

for all $(y, z) \in \mathbb{R}^{m+2p}$, entailing $g(y, 0) \equiv 0$. Let us make the extra assumption

$$
\bigwedge_{y \in \mathbb{R}^m} h(y, 0) = 0 \tag{7}
$$

which ensures that $\mathbb{T}^n \times \mathbb{R}^m \times \{0\} \subseteq \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ is a family of invariant tori. In the discussion to follow we will identify conditions on (6) that justify this assumption. Where (7) fails we ultimately encounter a quasi-periodic centre-saddle bifurcation. In the expansion

$$
\begin{aligned}
\dot{x} &= \omega(y) + \mathcal{O}(z) \\
\dot{y} &= \mathcal{O}(z) \\
\dot{z} &= \Omega(y) \cdot z + \mathcal{O}(z^2)
\end{aligned} \tag{8}
$$

the matrix $\Omega(y) \in \mathfrak{gl}(2p, \mathbb{R})$ is x–independent (like all terms on the right hand side), because of integrability the tori are in Floquet form. It is helpful to decouple the y–dependence in (8) from the dominant part and consider

$$
\begin{aligned}
\dot{x} &= \omega \\
\dot{y} &= 0 \\
\dot{z} &= \Omega z
\end{aligned} \tag{9}
$$

with parameters $\omega \in \mathbb{R}^n$ and Ω. Because of reversibility the latter matrix is an element of

$$
\mathfrak{gl}_-(2p, \mathbb{R}) := \left\{ \Omega \in \mathfrak{gl}(2p, \mathbb{R}) \;\middle|\; \Omega R = -R\Omega \right\}
$$

whence the spectrum of Ω consists of conjugate purely imaginary pairs $\pm \mathrm{i}\alpha$, symmetric real pairs $\pm\beta$, complex quartets $\pm\beta \pm \mathrm{i}\alpha$ and, if 0 is an eigenvalue, 0 with even algebraic multiplicity. Excluding the eigenvalue 0 enforces (7) straightforwardly; if $\det \Omega(y) \neq 0$ then $h(y, z(y)) \equiv 0$ with $z(y)$ given by the implicit mapping theorem allows for the translation $(y, z) \mapsto (y, z - z(y))$.

The hyperbolic part may be dealt with by means of reduction to a centre manifold, so let us further assume that Ω is elliptic. As Ω and R anti-commute we can find a simultaneous block-diagonalization

$$
\Omega = \operatorname{diag}(\alpha_1 J_2, \ldots, \alpha_p J_2) \;, \quad R = \operatorname{diag}(R_2, \ldots, R_2)
$$

with

$$
J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{and} \qquad R_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;,
$$

for multiple frequencies see below.

Next to internal resonances we also have to avoid normal-internal resonances and replace (3) by

$$\bigwedge_{\substack{k\in\mathbb{Z}^n\\ k\neq 0}} \bigwedge_{\substack{\ell\in\mathbb{Z}^p\\ |\ell|\leq 2}} |2\pi\langle k \mid \omega\rangle + \langle \ell \mid \alpha\rangle| \geq \frac{\gamma}{|k|^\tau} . \tag{10}$$

The (extended) frequency mapping

$$(\omega, \alpha) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathbb{R}^p \tag{11}$$

allows to pull back the geometry defined by (10) into phase space.

Theorem 2 [9, 26, 10, 12, 7, 3] *BHT non-degeneracy implies quasi-periodic stability.*

The non-degeneracy condition of Broer, Huitema and Takens formulated in [11] requires the frequency mapping to be a submersion and all eigenvalues to be simple. The resulting quasi-periodic stability is rather strong as Diophantine tori not only persist under small reversible perturbations, but furthermore retain their normal linear behaviour. Similar to theorem 1 the submersivity of (11) can be relaxed to $(\omega, \alpha)(\mathbb{R}^m)$ being sufficiently curved to have a large intersection with $(\mathbb{R}^n \times \mathbb{R}^p)_{\gamma,\tau}$. This allows for smaller dimension m and in particular makes the important case $m = n$ accessible.

In [12, 7] the requirement that all eigenvalues be simple is dropped and tori $\mathbb{T}^n \times \{y_0\} \times \{0\}$ are considered for which $\Omega_0 := \Omega(y_0)$ has multiple eigenvalues. To still ensure (7) the condition $\det \Omega_0 \neq 0$ is retained, while (11) is replaced by the (amended) frequency mapping

$$(\omega, \Omega) : \mathbb{R}^m \longrightarrow \mathbb{R}^n \times \mathfrak{gl}_-(2p, \mathbb{R}) . \tag{12}$$

At points $y_1 \in \mathbb{R}^m$ where $\Omega_1 = \Omega(y_1)$ has only simple eigenvalues the submersivity of (11) makes (12) transverse to the orbit of Ω_1 under the adjoint action $\Omega \mapsto T\Omega T^{-1}$ of

$$\mathrm{GL}_+(2p, \mathbb{R}) := \left\{ T \in \mathrm{GL}(2p, \mathbb{R}) \;\middle|\; TR = RT \right\} .$$

This latter property is imposed on (12) also at y_0, whence the amended frequency mapping provides a versal unfolding of the Jordan normal form of Ω_0.

The resulting quasi-periodic stability still exerts full control on the normal linear behaviour of the persisting tori. The complete versal unfolding of Ω_0 is present in the perturbed system, with the persisting tori parametrised by large Cantor sets. Compare this with the approach in [32, 30] where the tori themselves are shown to be persistent under weaker conditions, but all control on the normal linear behaviour is lost. As exemplified in [2] such a control is needed if one wants to address the bifurcations resulting from multiple frequencies.

To address bifurcations resulting from vanishing frequencies a non-degeneracy condition relaxing $\det \Omega_0 \neq 0$ is genuinely helpful. The formulation in [11] similarly already anticipated the (dissipative) frequency-halving bifurcation treated in [1]. Writing (9) in vector field notation $N = \omega\partial_x + \Omega z\partial_z$, the Lie bracket with a constant vector field $\beta\partial_z$ reads as

$$[N, \beta\partial_z] \;\; = \;\; -\Omega\beta\,\partial_z$$

and vanishes if and only if $\beta \in \ker\Omega$. Invertibility of the adjoint operator

$$\operatorname{ad} N \;:\; \beta\partial_z \;\; \mapsto \;\; [N, \beta\partial_z]$$

ensures that the constant part of any perturbing vector field can be transformed away. In important cases it is possible to first restrict $\operatorname{ad} N$ to a subspace $\mathcal{B}$ of the space of constant vector fields. For instance, where zero eigenvalues result from a lift to a 2–fold covering space the lifted vector fields are equivariant with respect to the deck transformation and the resulting subspace $\mathcal{B}$ does not contain the corresponding (generalized) zero eigendirections.

In the present reversible setting one can always restrict to the subspace $\mathcal{B}^+$ of constant vector fields that are equivariant with respect to the reversor G (not to be confounded with the subspace $\mathcal{B}^-$ of reversible constant vector fields). The definition of BHT non-degeneracy adopted in [3] is a condition on the torus $\mathbb{T}^n \times \{y_0\} \times \{0\}$ taking the form that

$$\ker\Omega(y_0) \;\cap\; \mathcal{B}^+ \;\; = \;\; \{0\} \tag{13}$$

and that (12) be at y_0 transverse to the product

$$\{\omega(y_0)\} \times \mathrm{GL}_+(2p, \mathbb{R})(\Omega_0) \;\; \subseteq \;\; \mathbb{R}^n \times \mathfrak{gl}_-(2p, \mathbb{R})$$

of the singleton with internal frequency vector $\omega(y_0)$ and the $\mathrm{GL}_+(2p, \mathbb{R})$–orbit of Ω_0.

The resulting quasi-periodic stability again exerts full control on the normal linear behaviour of the persisting tori. The amended frequency mapping (12) now also provides a versal unfolding of the nilpotent part of Ω_0. In the next section we use this to address the resulting bifurcations.

3 Pitchfork bifurcations

The frequency-halving bifurcation leads to a quasi-periodic pitchfork bifurcation on a 2–fold covering space. We therefore first treat the latter in its own right. Note that (13) boils down to

$$\ker\Omega_0 \;\; \subseteq \;\; \mathrm{Fix}(-R)$$

because of $\Omega_0 R = -R\Omega_0$. Here we consider the simplest case of geometric multiplicity 1 and algebraic multiplicity 2 ; for definiteness we furthermore restrict to $p = 1$ altogether, thereby dispensing with additional normal frequencies. Splitting $z = (u, v)$ such that $Rz = (-u, v)$, the reversor takes the form

$$G(x, y, u, v) \;\; = \;\; (-x, y, -u, v) \;\; .$$

Theorem 2, or its Corollary 6 in [3], yields the persistence of the whole family $\{u = v = 0\}$ of invariant tori (parametrised by Diophantine ω), in particular of the parabolic tori

$$\mathbb{T}^n \times \{y_0\} \times \{(0, 0)\} \;\; \subseteq \;\; \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^2 \;\; .$$

To obtain information on the tori that are generated by the bifurcation we need non-linear terms from (8). After a normalization with respect to (9), cf. [1, 2], the lowest order terms read as

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= 0 \\ \dot{u} &= av \\ \dot{v} &= (\lambda - bu^2 + cv)u \end{aligned} \tag{14}$$

from which we compute the position $u_0 = \pm\sqrt{\lambda/b}$, $v_0 = 0$ of the tori bifurcating off from $(y, u, v) = (y_0, 0, 0)$. Here λ is a bifurcation parameter, similarly to ω and Ω we will eventually have $\lambda = \lambda(y)$ with $\lambda(y_0) = 0$ whence the additional tori

$$\mathbb{T}^n \times \{y\} \times \{(u(y), 0)\} \tag{15}$$

exist for $\lambda(y) > 0$ in case $b > 0$ and for $\lambda(y) < 0$ in case $b < 0$.

Theorem 3 *Let X be a real analytic vector field on $\mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^2$ for which the torus $\mathbb{T}^n \times \{y_0\} \times \{(0, 0)\}$ is parabolic and the coefficient functions $\lambda, a, b, c : \mathbb{R}^m \longrightarrow \mathbb{R}$ in the normal form* (14) *satisfy $\lambda(y_0) = 0$, $D\lambda(y_0) \neq 0$, $a(y_0) \neq 0$, $b(y_0) \neq 0$ and, if $a(y_0) \cdot b(y_0) > 0$, also $c(y_0) \neq 0$. Then X undergoes a quasi-periodic reversible pitchfork bifurcation.*

The case of positive $a(y_0) \cdot b(y_0)$ is the supercritical case and where $a(y_0)$ and $b(y_0)$ have different sign a subcritical quasi-periodic reversible pitchfork bifurcation takes place. See also [18, 28].

Proof. We still have to show that the tori (15) persist under a small perturbation of (14). As only the relative sign between $a(y_0)$ and $b(y_0)$ is relevant we fix $\operatorname{sgn} b(y_0) = +1$. Concentrating on the plus sign $u(y) = +\sqrt{\lambda(y)/b(y)}$ we translate the torus by means of

$$(x, y, u, v) \mapsto (x, y, u - \sqrt{\frac{\lambda(y)}{b(y)}}, v)$$

to the origin, with dominant part

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= 0 \\ \dot{u} &= av \\ \dot{v} &= -2\lambda u + c\sqrt{\frac{\lambda}{b}}\, v \end{aligned}$$

and apply the scaling

$$(x, y, u, v, \omega, t) \mapsto (x, \lambda^{-\frac{5}{2}} y, \lambda^{-1} u, \lambda^{-\frac{3}{2}} v, \lambda^{-\frac{1}{2}} \omega, \lambda^{\frac{1}{2}} t)$$

to obtain the normal linear behaviour

$$\begin{pmatrix} 0 & a \\ -2 & c/\sqrt{b} \end{pmatrix} .$$

In the subcritical case $\operatorname{sgn} ab = \operatorname{sgn} a = -1$ this is of saddle type and the image torus under G (the one with negative u–co-ordinate) is of saddle type as well. In the supercritical case of positive a the normal linear behaviour is attractive if $c < 0$ and repulsive if $c > 0$, the image torus under G is then repulsive or attractive, respectively. In any case we can apply Corollary 4.2 of [11] to obtain quasi-periodic stability. □

Thus, in the supercritical case an attractor-repellor pair of tori bifurcates off from the original tori, which thereby turn from elliptic to hyperbolic. Note that for any $\delta > 0$ the union

$$\mathbb{T}^n \times \left\{ (y,u,v) \in \mathbb{R}^{m+2} \;\middle|\; \lambda(y) > \delta b(y)\,,\; u = \pm\sqrt{\frac{\lambda(y)}{b(y)}}\,,\; v = 0 \right\} \tag{16}$$

is an attractive/repulsive manifold and persists as such because of Theorem 4.1 in [17]. This carries *mutatis mutandis* over to the normally hyperbolic manifold (16) in the subcritical case. However, this does not apply to the individual tori as there is no attraction/repulsion in the y–direction.

In particular, the tori bifurcating off from $\mathrm{Fix}(-R)$ do not completely behave as in a dissipative system. There, a single Diophantine torus may form a normally hyperbolic manifold and thus persist by Theorem 4.1 of [17] a small perturbation that makes the frequency vector resonant. This leads to full families $\mathbb{T}^n \times \mathbb{R}^m$ of tori as long as the attraction/repulsion normal to the tori (the $\mathbb{R}^m$–direction represents external parameters and does not carry normal dynamics) dominates the attraction/repulsion within the phase-locked tori. Such a “fattening by hyperbolicity” as described in [1] does not take place in the present reversible setting.

The lifted vector field on a 2–fold covering is equivariant with respect to the deck transformation, which in adapted x–co-ordinates takes the form

$$F \;:\; (x,y,u,v) \;\longmapsto\; (x_1 - \tfrac{1}{2}, x_2, \ldots, x_n, y, -u, -v)$$

and makes the vector field reversible with respect to a second reversor $H := F \circ G$. In the normal form this enforces $c \equiv 0$ and (14) turns into

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= 0 \\ \dot{u} &= av \\ \dot{v} &= \lambda u - bu^3 \;. \end{aligned} \tag{17}$$

In the supercritical case this reflects that the additional tori (15) are invariant under H and must therefore be elliptic.

Theorem 4 *Let X be a real analytic vector field on $\mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^2$ that is reversible with respect to both G and H. Assume that the torus $\mathbb{T}^n \times \{y_0\} \times \{(0,0)\}$ is parabolic with coefficient functions $\lambda, a, b : \mathbb{R}^m \longrightarrow \mathbb{R}$ in the normal form* (17) *satisfying $\lambda(y_0) = 0$, $D\lambda(y_0) \neq 0$, $a(y_0) \neq 0$ and $b(y_0) \neq 0$. Then at $y = y_0$ a pair of tori* (15) *bifurcates off from $\mathbb{T}^n \times \mathbb{R}^m \times \{(0,0)\}$, elliptic in the supercritical case $a(y_0) \cdot b(y_0) > 0$ and hyperbolic in the subcritical case $a(y_0) \cdot b(y_0) < 0$.*

What takes place is a quasi-periodic $\mathbb{Z}_2$–equivariant reversible pitchfork bifurcation, where the symmetry group $\mathbb{Z}_2$ is generated by $F = G \circ H$. See also [18, 28].

Proof. We follow the proof of theorem 3, but apply at the end the Corollary to the Main Theorem in [9] to obtain quasi-periodic stability of the elliptic or hyperbolic tori. □

Under the projection of the 2–fold covering to the base space the original tori are covered twice by $\mathbb{T}^n \times \{y\} \times \{(0,0)\}$ while both tori (15) are mapped bijectively to the same additional torus in the phase space. Hence, this torus has the first frequency divided by 2 as compared to the torus it bifurcates off from.

Corollary 5 *The frequency-halving bifurcation of an integrable reversible vector field persists under small perturbations.* □

4 The quasi-periodic centre-saddle bifurcation

Parabolic tori have normal linear behaviour governed by $\Omega_0 = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right)$ with $a \neq 0$ and if the reversor takes the form

$$G(x, y, u, v) \quad = \quad (-x, y, u, -v)$$

then $\ker \Omega_0 \subseteq \mathrm{Fix}(R)$ whence $\mathrm{ad}\, N : \mathcal{B}^+ \longrightarrow \mathcal{B}^-$ is not invertible. In this situation the assumption (7) is no longer justified, correspondingly the resulting bifurcations lead to more than just changes in the normal linear behaviour. In the simplest case, of co-dimension 1, elliptic and hyperbolic tori annihilate each other when meeting at the parabolic tori. The integrable normal form of this quasi-periodic centre-saddle bifurcation reads as

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= 0 \\ \dot{u} &= av \\ \dot{v} &= \lambda - bu^2 \end{aligned}$$

with tori at $u_0 = \pm\sqrt{\lambda/b}$, $v_0 = 0$. Using the scaling

$$(u, v, \lambda) \quad \mapsto \quad \left(\frac{u}{ab}, \frac{v}{a^2 b}, \frac{\lambda}{a^2 b}\right)$$

we may restrict to $a = b = 1$ whence invariant n–tori only exist for $\lambda \geq 0$ with elliptic tori at $u_0 = +\sqrt{\lambda} > 0$ and hyperbolic tori at $u_0 = -\sqrt{\lambda} < 0$.

Theorem 6 *Let the vector field X on $\mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^2$ be an ε–small reversible perturbation of $\omega\partial_x + v\partial_u + (\lambda - u^2)\partial_v$ depending on parameters $\lambda \in \mathbb{R}$ and $\omega \in \Sigma \subseteq \mathbb{R}^n$. Then there are mappings $\widehat{\omega}, \widehat{\Omega}, \phi$ in the variables $(u, \lambda) \in \mathbb{R} \times \Lambda$, with Λ an open interval around 0, satisfying*

$$\begin{aligned} \widehat{\omega}(u, \lambda) &= \omega + \mathcal{O}(\varepsilon) \in \mathbb{R}^n \\ \widehat{\Omega}(u, \lambda) &= \begin{pmatrix} 0 & 1 \\ -2u & 0 \end{pmatrix} + \mathcal{O}(\varepsilon) \in \mathfrak{gl}_-(2, \mathbb{R}) \\ \phi(u, \lambda) &= \lambda - u^2 + \mathcal{O}(u^3) + \mathcal{O}(\varepsilon) \in \mathbb{R} \end{aligned}$$

and having the following property. The solutions of $\phi(u,\lambda) = 0$ *for which* $(\widehat{\omega}(u,\lambda), \widehat{\Omega}(u,\lambda))$ *is Diophantine determine invariant tori of* X *that have normal linear behaviour conjugate to* $\widehat{\omega}(u,\lambda)\,\partial_x + \widehat{\Omega}(u,\lambda)\,z\,\partial_z$.

Thus, also X undergoes a quasi-periodic centre-saddle bifurcation as, under variation of λ, elliptic tori parametrised by a Cantor set defined by (10) and hyperbolic tori parametrised by a Cantor set defined by (3) meet at parabolic tori parametrised by a Cantor set defined by (3) and vanish. This answers a conjecture in [14] to the positive.

Proof. This follows directly from the conclusion of Theorem 3.1 in [29]. The proof of that result uses (the Main) Theorem 2.3 of [29] which is formulated in terms of an admissible pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras. As worked out in Section 3.2 of [9], in the present reversible setting the Lie algebra $\mathfrak{h}$ of structure-preserving vector fields can be replaced by the vector space $\mathfrak{h}_-$ of reversible vector fields together with the Lie algebra $\mathfrak{h}_+$ of G–equivariant vector fields, while the rôle of the finite-dimensional Lie algebra $\mathfrak{g} < \mathfrak{h}$ is taken over by $\mathfrak{gl}_-(2,\mathbb{R})$ together with $\mathfrak{gl}_+(2,\mathbb{R})$. □

The crucial property is $[\mathfrak{h}_+, \mathfrak{h}_-] \subseteq \mathfrak{h}_-$ as this ensures that a vector field Y solving the homological equation is G–equivariant and thus generates a time–t–mapping Ψ that transforms reversible vector fields into reversible vector fields. See also [20, 27]. In this way theorems 1–4 can be deduced from Theorems 2.3 and 3.1 of [29] as well, see [5].

5 Conclusions

Invariant tori in Floquet form undergo a semi-local bifurcation of co-dimension 1 if there is a Floquet exponent on the imaginary axis with geometric multiplicity 1 and algebraic multiplicity 2. For a vanishing Floquet exponent this leads to the bifurcations treated in theorems 6 and 3 and corollary 5. The alternative is a double pair of elliptic eigenvalues in 1:1 resonance encountered during a quasi-periodic reversible Hopf bifurcation, see [2] for more details.

While the frequency-halving, quasi-periodic centre saddle and reversible Hopf bifurcations are in essence already of Hamiltonian nature, compare with [15], the quasi-periodic reversible pitchfork bifurcation gets replaced by its counterpart $c \equiv 0$ of theorem 4 if one considers reversible Hamiltonian systems. Similar differences occur in higher co-dimensions. For instance, vanishing Floquet exponents with algebraic and geometric multiplicity 2 lead to the systems

$$\begin{aligned} \dot{x} &= \omega \\ \dot{y} &= 0 \\ \dot{u} &= auv + \lambda_2 v \\ \dot{v} &= \lambda_1 - v^2 \pm u(u - \lambda_2) \end{aligned} \tag{18}$$

undergoing bifurcations of co-dimension 2, cf. [28, 13]. For $a = 2$ the reversible system (18) is Hamiltonian, and as long as the coefficient a is positive this Hamiltonian nature prevails. The cases with $a < 0$ are of essentially non-Hamiltonian nature.

Quasi-periodic bifurcations help to explain the dynamics in the gaps opened by resonances. For instance, in a family of elliptic tori the normal-internal resonances

$$2\pi\langle k \mid \omega\rangle \;+\; \langle \ell \mid \alpha\rangle \;=\; 0 \tag{19}$$

with $0 \neq k \in \mathbb{Z}^n$ and $\ell \in \mathbb{Z}^p, |\ell| \leq 2$ lead to gaps left open by tori not satisfying (10). Crossing a single resonance (19) generically generates two bifurcations of co-dimension 1 where the elliptic tori born in the bifurcations lead to secondary gaps within gaps, see [4] for an in-depth analysis in the Hamiltonian context.

The following example shows that quasi-periodic bifurcations are not only possible and furthermore unavoidable where one perturbs from integrable systems already displaying quasi-periodic bifurcations, but in fact occur for generic perturbations of integrable systems satisfying the mild conditions of theorem 1. The integrable system is fibrated by invariant tori $\mathbb{T}^n \times \{y\}$, $y \in \mathbb{R}^m$ and the question is what happens in the gap left open by theorem 1 in case of a single resonance

$$\langle k \mid \omega\rangle \;=\; 0 \;. \tag{20}$$

In suitable co-ordinates this resonance reads as $\omega_n = 0$ and one can split off the last components $u := x_n \in \mathbb{T}$ and $v := y_m \in \mathbb{R}$ to describe the normal behaviour of $(n-1)$–tori

$$\mathbb{T}^{n-1} \times \{(u^0, y_1, \ldots, y_{m-1}, v^0)\} \;\subseteq\; \mathbb{T}^{n-1} \times \mathbb{T} \times \mathbb{R}^{m-1} \times \mathbb{R} \;.$$

For the unperturbed system this normal behaviour is trivial as the $(n-1)$–tori fibrate the invariant n–tori $\mathbb{T}^n \times \{y\}$. Averaging the perturbed system along the $(n-1)$–tori we obtain an x–independent normal form

$$\begin{aligned} \dot{x}_i &= \omega_i \;, \quad i = 1, \ldots, n \\ \dot{y}_i &= 0 \;, \quad i = 1, \ldots, m \\ \dot{u} &= f_y(u, v) \\ \dot{v} &= g_y(u, v) \end{aligned}$$

where solutions (u^0, v^0) of $f = g = 0$ correspond to invariant $(n-1)$–tori and variation of y may lead to bifurcations of these. The genericity considerations of [6] apply *mutatis mutandis*, in particular it is generic for a single system $\dot{u} = f(u, v)$, $\dot{v} = g(u, v)$ to have only equilibria with simple eigenvalues, but a whole family of systems depending on $y \in \mathbb{R}^{m-1}$ may encounter bifurcations of co-dimension up to $m - 1$ (of which the bifurcations of co-dimension $\leq m - 2$ subsequently persist the passage from the normal form to the original perturbation on large Cantor sets).

In case of multiple internal resonances (20) one can similarly split off p pairs from x and y to let $(u, v) \in \mathbb{R}^{2p}$ describe the normal behaviour of resulting $(n - p)$–tori. Where the number p of resonances exceeds m one has to work in what is termed the reversible context 2 in [10, 27].

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