Homework 5

(*i*) Show that the space of axially symmetric vector fields is generated by the three vector fields

$$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \ , \quad x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial z} \ ,$$

meaning that the most general $\mathbb{S}^1$–equivariant vector field has the form

$$f(\tau, z)(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) \ + \ g(\tau, z)(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}) \ + \ h(\tau, z)\frac{\partial}{\partial z} \tag{1}$$

where $\tau = \frac{1}{2}(x^2 + y^2)$.

(*ii*) Show that for (1) to be volume-preserving, the coefficient functions have to satisfy

$$2\frac{\partial \tau g}{\partial \tau} \ + \ \frac{\partial h}{\partial z} \ = \ 0 \ . \tag{2}$$

(*iii*) In the invariants τ and z the vector field (1) reduces to

$$\dot{\tau} \ = \ 2\tau g(\tau, z) \tag{3a}$$

$$\dot{z} \ = \ h(\tau, z) \tag{3b}$$

whence the line $\{\tau = 0\}$ is always invariant — as expected from the $\mathbb{S}^1$–symmetry. Use (2) to show that the equations of motion (3) are Hamiltonian with Hamiltonian function

$$H(\tau, z) \ = \ \int_0^z 2\tau g(\tau, \tilde{z}) \, \mathrm{d}\tilde{z} \ - \ \int_0^\tau h(\tilde{\tau}, 0) \, \mathrm{d}\tilde{\tau} \ .$$