Matrices depending on parameters

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Goal

How can we define a simple normal form for not diagonizable matrices? Expecially for matrices close to each other.

Deformations

A deformation of a matrix $A_0 \in \mathbb{C}^{n\times n}$ is a matrix $A(\gamma) \in \mathbb{C}^{n\times n}$ with:

- enteries that are power series of variables $\gamma_i \in \mathbb{C}$
- variables γ_i close to zero, convergent in the neighbourhood of $\gamma = 0$ with $A(0) = A_0$

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$$\text{Example: } A(\gamma) = \begin{pmatrix} 1+\gamma_1 & 3+(1-\gamma_2)^2 \\ \gamma_3\gamma_1 & 5+\gamma_4^3 \end{pmatrix}$$

Equivalent deformations

Two deformations $A(\gamma)$ and $B(\gamma)$ of matrix A_0 are **equivalent** if there exists a deformation $C(\gamma)$ of the identity matrix ($C(0) = I_n$) such that:

$$A(\gamma) = C(\gamma)B(\gamma)C(\gamma)^{-1},$$

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Mapping in parameter space

Define a mapping $\varphi : C^l \to C^k$ close to zero, which is convergent in the neighbourhood of zero with $\varphi(0) = 0$. Such that φ is a mapping of the parameter space $\{\mu\}$ to $\{\gamma\}$, and $A(\gamma) = A(\varphi(\mu))$.

Versal deformation

A **versal** deformation of a matrix A_0 is a deformation which is *equivalent* to every other deformation of A_0 under a suitable *change of parameters*:

$$B(\mu) = C(\mu)A(\varphi(\mu))C(\mu)^{-1}, \text{ for every } B(0) = A_0, \\ \text{with } C(0) = I_n, \ \varphi(0) = 0.$$

The deformation is **universal** if the change of parameters φ is unique for each $B(\mu)$.

Goal: Find the simplest versal deformation for matrices A_0, with the least number of parameters (**miniversal**)

Transversality

Consider a smooth mapping $A : \gamma \to M$ where $M \subset \mathbb{C}^{n \times n}$,
$N \subset M$ and let γ be a point in Λ such that $A(\gamma) \in N$.
Then the mapping A is called **transversal** to N at γ if the tangent space to M at $A(\gamma)$ is the direct sum of tangent space of N at $A(\gamma)$ and the tangent space of the mapping A:

$$TM_{A(\gamma)} = A_* T\Lambda_\gamma \oplus TN_{A(\gamma)}$$

Orbit

Consider a set $M = \mathbb{C}^{n\times n}$ and the group $G = \{\det(c) \neq 0 | c \in \mathbb{C}^{n\times n}\}$, then the **orbit** of $m \in M$ is given by the set $G(m) = \{gmg^{-1} | g \in G\}$.

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Lemma 1

A deformation $A(\gamma)$ is **versal** $\Leftrightarrow$ *the mapping A is* **transversal to the orbit** *of A_0 at $\gamma = 0$.*

Universality of a sylvester family

A sylvester family:

$$A(\alpha) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ . & . & . & . & . \\ & & & 0 & 1 \\ \alpha_1 & \alpha_2 & & \dots & \alpha_n \end{pmatrix}$$

defines a universal deformation of each of its matrices.

Minimal versal deformation for A_0 in Jordan normal form

A matrix A_0 in Jordan normal form has a versal deformation of the form $A_0 + B(\alpha)$ with for each Jordan block i: $B_i(\alpha)$ with

$$\text{non-zeros at} \left(\begin{array}{ccc|cc|c} \alpha_1 & \dots & \alpha_{n_1} & \dots & \alpha_{n_1+n_2} & \dots \\ \hline \vdots & & & & & \\ \vdots & & & \dots & \dots & \dots \\ \hline \vdots & & & \vdots & & \dots \end{array} \right)$$

and minimal number of parameters $d = \sum_{j=1}^{N_i} (2j-1) n_j$, for i´th Jordan block of length N_i with eigenvalue λ_i of orders $n_1 \geq \cdots \geq n_{N_i}$.