Period-Doubling Bifurcation

Essay

Introduction to Nonlinear Dynamical Systems
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1 Introduction

Discrete-time dynamics are often studied by iterating a map that depends on a parameter. In this essay we consider a one-dimensional family

$$x \mapsto f(x, \alpha), \qquad x \in \mathbb{R},\ \alpha \in \mathbb{R},$$

and ask how the orbit structure changes when α varies.

Our focus is the *period-doubling (flip) bifurcation*. Here a fixed point reaches the stability boundary at multiplier -1: as α passes through the bifurcation value, the fixed point loses stability and a nearby periodic orbit of period two appears.

The essay follows the four parts of the assignment. We first state the flip condition for a fixed point and introduce the local expansion near the bifurcation. Next, under genericity assumptions, we prove existence and uniqueness of the emerging 2-cycle and determine its stability via the second iterate. We then show that the dynamics are locally topologically equivalent to the normal form

$$\xi \mapsto -(1 + \beta)\xi \pm \xi^3.$$

Finally, we apply the results to the logistic map and compute the first flip point and the stability range of the resulting period-2 orbit.

Throughout, the keys are the multiplier crossing -1, the cubic coefficient $H(0) \neq 0$, and the use of f^2 to detect and study 2-cycles.

Contents

2 Period-doubling bifurcation

2.1 Q1: Setup and flip condition

We describe the period-doubling bifurcation of a fixed point in the smooth scalar mapping (1). Let f be C^r $(r \geq 3)$ and consider the one-parameter family

$$x \mapsto f(x, \alpha), \qquad x \in \mathbb{R},\ \alpha \in \mathbb{R}. \tag{1}$$

A fixed point x^* at $\alpha = \alpha_0$ satisfies $f(x^*, \alpha_0) = x^*$. Its (linear) stability is determined by the multiplier

$$\mu(\alpha) := \partial_x f(x^*(\alpha), \alpha),$$

where $x^*(\alpha)$ denotes the local continuation of the fixed point, which exists by the implicit function theorem applied to $g(x, \alpha) := f(x, \alpha) - x$, since at a flip point $\partial_x f(x^*, \alpha_0) = -1$ and hence

$$g_x(x^*, \alpha_0) = \partial_x f(x^*, \alpha_0) - 1 = -2 \neq 0.$$

Period-doubling (or *flip*) bifurcation occurs when the multiplier crosses -1 at $\alpha = \alpha_0$:

$$\mu(\alpha_0) = -1, \qquad \mu'(\alpha_0) \neq 0.$$

Thus the fixed point is nonhyperbolic at $\alpha = \alpha_0$, and its stability changes as α passes through α_0 (since stability requires $|\mu(\alpha)| < 1$).

To describe the local dynamics, shift the fixed point to the origin by a smooth parameter-dependent translation $x = x^*(\alpha) + \xi$. Then $\xi = 0$ is a fixed point for all nearby α, and the Taylor expansion has the form

$$\tilde{\xi} = \mu(\alpha)\xi + b(\alpha)\xi^2 + c(\alpha)\xi^3 + O(\xi^4).$$

2.2 Q2: Existence of a 2-cycle

We now prove that, for a generic one-parameter family of C^r $(r \geq 3)$ scalar maps (1), a period-doubling bifurcation at (x_0, α_0) produces a unique nearby 2-cycle, while the fixed point changes stability.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be C^r with $r \geq 3$ and consider the family

$$x \mapsto f(x, \alpha).$$

Assume that at (x_0, α_0) there is a fixed point,

$$f(x_0, \alpha_0) = x_0, \tag{2}$$

with multiplier

$$\mu_0 := f_x(x_0, \alpha_0) = -1.$$

We first state the relevant genericity conditions:

(G1) (*Persistence of the fixed point*) $f_x(x_0, \alpha_0) - 1 \neq 0$ (in fact $f_x(x_0, \alpha_0) - 1 = -2$ since $f_x(x_0, \alpha_0) = -1$). Hence, by the Implicit Function Theorem, there is a unique C^r-smooth fixed point branch $x_*(\alpha)$ near α_0 with $x_*(\alpha_0) = x_0$.

(G2) (*Flip condition*) $f_x(x_0, \alpha_0) = -1$.

(G3) (*Transversality*) Along the fixed point branch, the multiplier crosses -1 with nonzero speed:

$$\frac{d}{d\alpha} f_x(x_*(\alpha), \alpha)\Big|_{\alpha=\alpha_0} \neq 0.$$

(G4) (*Nondegeneracy of the cubic normal form coefficient*) After shifting the fixed point to the origin (so that $x_*(\alpha) \equiv 0$), we write the Taylor expansion

$$\tilde{x} = \mu(\alpha)x + b(\alpha)x^2 + c(\alpha)x^3 + O(x^4), \qquad \mu(\alpha_0) = -1. \tag{3}$$

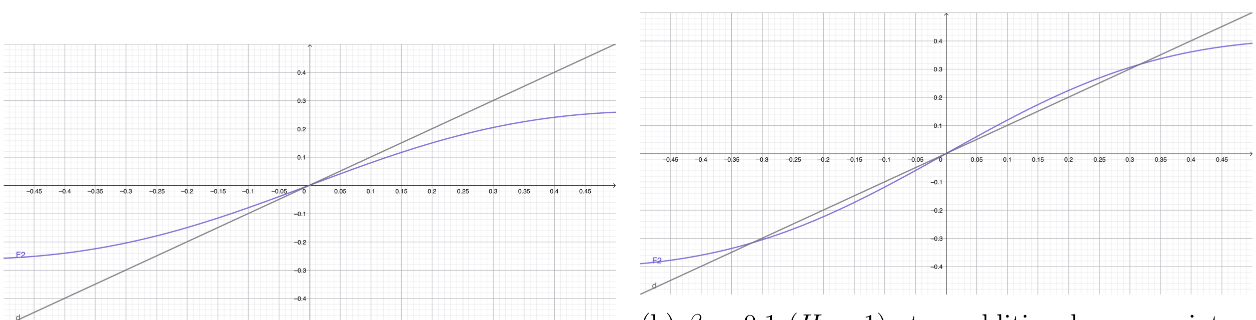
By Lemma 7.9 in the reader, there exists a smooth change of variables that removes the quadratic term, yielding the reduced form

$$\tilde{y} = \mu(\alpha)y + H(\alpha)y^3 + O(y^4),$$

where H is smooth and

$$H(\alpha_0) = c(\alpha_0) + b(\alpha_0)^2 \neq 0.$$

We study period-2 orbits as nontrivial fixed points of the second iterate, using the cubic reduction from Lemma 7.9. Figure 1 visualizes the key step in the proof: a period-2 orbit corresponds to nonzero solutions of $F_\beta^2(y) = y$, i.e. intersections of the graph of F_β^2 with with the diagonal $\tilde{y} = y$. Note that the intersection $y = 0$ corresponds to the fixed point and does not give a period-2 orbit. A genuine period-2 orbit corresponds to nonzero fixed points of F_β^2, and for $\beta > 0$ these occur as a symmetric pair $y_\pm(\beta)$ (with $y_+(\beta) = -y_-(\beta)$), forming the 2-cycle $y_+ \mapsto y_- \mapsto y_+$.



(a) $\beta = -0.1$ ($H = 1$): only the intersection at $y = 0$.

(b) $\beta = 0.1$ ($H = 1$): two additional nonzero intersections $y_\pm(\beta)$.

Figure 1: Intersections of F_β^2 with the diagonal for the cubic truncation $F_\beta(y) = -(1+\beta)y + Hy^3$.

Proof. Using (G1), shift $x \mapsto x - x_*(\alpha)$ so that 0 is a fixed point for all α near α_0. Then we have an expansion of the form (3). By Lemma 7.9, there is a smooth local change of variables $x = y + \delta(\alpha)y^2$ that removes the quadratic term and gives us

$$y \mapsto \tilde{y} = \mu(\alpha)y + H(\alpha)y^3 + O(y^4), \qquad H(\alpha_0) = c(\alpha_0) + b(\alpha_0)^2 \neq 0. \tag{4}$$

We reparametrize the parameter by $\beta = \beta(\alpha)$ as in (G3), so that

$$\mu(\beta) = -(1+\beta), \qquad \beta(\alpha_0) = 0, \qquad \beta'(\alpha_0) \neq 0.$$

Then (4) becomes

$$y \mapsto F_\beta(y) = -(1+\beta)y + H(\beta)y^3 + R_1(y,\beta), \tag{5}$$

where $R_1(y,\beta) = O(y^4)$ is C^r-smooth and $H(0) \neq 0$.

Since $H(\beta) = H(0) + O(\beta)$, we absorb $(H(\beta) - H(0))y^3$ into the remainder and write

$$F_\beta(y) = -(1+\beta)y + H(0)y^3 + R(y,\beta), \qquad R(y,\beta) = O(|\beta|\, y^3 + y^4).$$

The fixed point is $y = 0$ for all β, and its multiplier equals $\mu(\beta) = -(1+\beta)$. Hence $|\,\mu(\beta)\,| < 1$ for $\beta < 0$ and $|\,\mu(\beta)\,| > 1$ for $\beta > 0$. So the fixed point loses (or gains) stability as β passes through 0.

A period-2 orbit corresponds to a nontrivial fixed point of the second iterate:

$$F_\beta^2(y) = y, \qquad y \neq 0.$$

Write $a := 1 + \beta$. For $|y|$ and $|\beta|$ small we have, from (5) and the remainder estimate,

$$F_\beta(y) = -ay + H(0)y^3 + O(|\beta|\, y^3 + y^4).$$

Composing once more,

$$F_\beta^2(y) = F_\beta\big(-ay + H(0)y^3 + O(|\beta| y^3 + y^4)\big),$$

and using again the same expansion for $F_\beta(\cdot)$ gives

$$\begin{aligned} F_\beta^2(y) &= -a\Big(-ay + H(0)y^3\Big) + H(0)\big(-ay\big)^3 + O(|\beta|\, y^3 + y^5) \\ &= a^2 y - aH(0)y^3 - a^3 H(0)y^3 + O(|\beta|\, y^3 + y^5) \\ &= a^2 y - H(0)(a + a^3)y^3 + O(|\beta|\, y^3 + y^5). \end{aligned}$$

Since $a = 1 + \beta$, we have $a + a^3 = 2 + O(\beta)$ and hence (absorbing $O(|\beta|\, y^3)$ into the remainder)

$$F_\beta^2(y) = a^2 y - 2H(0)y^3 + O(|\beta|\, y^3 + y^5). \tag{6}$$

We define

$$G(y,\beta) := \frac{F_\beta^2(y) - y}{y}$$

for $y \neq 0$. Using (6) we have

$$G(y,\beta) = (a^2 - 1) - 2H(0)y^2 + o(|\beta| + y^2) \qquad \text{as } (y,\beta) \to (0,0). \tag{7}$$

Now set $u := y^2 \geq 0$ and consider

$$\Phi(u,\beta) := (a^2 - 1) - 2H(0)u + o(|\beta| + u).$$

We have $\Phi(0,0) = 0$ and $\partial_u \Phi(0,0) = -2H(0) \neq 0$ by (G4). Hence, by the Implicit Function Theorem, there exists $\varepsilon > 0$ and a unique smooth function $u(\beta)$ for $|\beta| < \varepsilon$ such that

$$\Phi(u(\beta),\beta) = 0, \qquad u(0) = 0,$$

and moreover

$$u(\beta) = \frac{a^2 - 1}{2H(0)} + o(\beta) = \frac{\beta}{H(0)} + o(\beta) \qquad (\beta \to 0). \tag{8}$$

Therefore:

- if $\beta/H(0) > 0$ and $|\beta|$ is small, then $u(\beta) > 0$ and (7) has exactly two small solutions

$$y_\pm(\beta) = \pm\sqrt{u(\beta)}.$$

- if $\beta/H(0) < 0$ and $|\beta|$ is small, then $u(\beta) < 0$ and there are no nonzero solutions near 0.

Finally, for $|\beta|$ small the map F_β has exactly one fixed point near 0, namely $y = 0$ (because $F_\beta(y) = y$ implies $(\mu(\beta) - 1)y + O(y^2) = 0$ and $\mu(\beta) - 1 \approx -2$). Hence any nonzero solution of $F_\beta^2(y) = y$ is not a fixed point of F_β, so $y_\pm(\beta)$ belong to a genuine 2-cycle. Moreover, the two points $y_+(\beta)$ and $y_-(\beta)$ form a single 2-cycle, because they are the only two nontrivial fixed points of F_β^2 in a neighbourhood. Thus, in the generic case there is exactly *one* small 2-cycle created (or destroyed) at $\beta = 0$.

□

Stability of the 2-cycle The multiplier of the 2-cycle equals $(F_\beta^2)'(y_\pm(\beta))$. Using (6) and (8) gives us

$$(F_\beta^2)'(y_\pm(\beta)) = a^2 - 6H(0)u(\beta) + o(|\beta|) = 1 - 4\beta + o(\beta).$$

Hence the 2-cycle is stable for $\beta > 0$ small and unstable for $\beta < 0$ small. Combining this with the existence condition $\beta/H(0) > 0$ gives us:

- $H(0) > 0$ (supercritical): for $\beta > 0$ a unique *stable* 2-cycle appears while the fixed point becomes unstable;
- $H(0) < 0$ (subcritical): for $\beta < 0$ a unique *unstable* 2-cycle exists while the fixed point is still stable.

2.3 Q3: Topological equivalence to the normal form

We now show that, near a generic period-doubling bifurcation of the family (1), the local dynamics is topologically equivalent to the normal form

$$\xi \mapsto -(1 + \beta)\xi \pm \xi^3. \tag{9}$$

Topological equivalence. Two maps f, g are (locally) topologically equivalent near their fixed points if there exist neighbourhoods U, V and a homeomorphism $h : U \to V$ such that

$$h \circ f = g \circ h \quad \text{on } U.$$

For a parameter family we require a family of such homeomorphisms h_β for $|\beta|$ small (with $h_\beta(0) = 0$), so that the local orbit structure is preserved for each nearby parameter value.

Proof. Starting from the Taylor expansion after shifting the fixed point to the origin (as in (3)), Lemma 7.9 gives a smooth near-identity change of variables $x = y + \delta(\alpha)y^2$ that removes the quadratic term. Since this is just a smooth change of coordinates, it does not affect the local orbit structure; it only simplifies the map. Thus the first relevant nonlinear term is the cubic term with coefficient $H(\alpha)$.

This yields the reduced family

$$y \mapsto \tilde{y} = \mu(\alpha)y + H(\alpha)y^3 + O(y^4),$$

with $H(\alpha_0) \neq 0$ (see (4)). By the transversality condition (G3), we can reparametrize by $\beta = \beta(\alpha)$ so that $\mu(\beta) = -(1+\beta)$ and $\beta(\alpha_0) = 0$, which gives the form (5):

$$y \mapsto F_\beta(y) = -(1+\beta)y + H(\beta)y^3 + R_1(y,\beta), \qquad R_1(y,\beta) = O(y^4).$$

Since $H(\beta) = H(0) + O(\beta)$ and $H(0) \neq 0$, we absorb $(H(\beta) - H(0))y^3$ into the remainder and write

$$F_\beta(y) = -(1+\beta)y + H(0)y^3 + R(y,\beta), \qquad R(y,\beta) = O(|\beta|\, y^3 + y^4).$$

By Theorem 5.26 in Lecture 10, higher-order terms do not change the local orbit structure. Hence the map

$$F_\beta(y) = -(1+\beta)y + H(0)y^3 + R(y,\beta), \qquad R(y,\beta) = O(|\beta|\, y^3 + y^4),$$

is locally topologically equivalent near 0 to its cubic truncation

$$y \mapsto -(1+\beta)y + H(0)y^3. \tag{10}$$

Since $H(0) \neq 0$, define the linear rescaling

$$\xi = \sqrt{|H(0)|}\, y, \qquad \tilde{\xi} = \sqrt{|H(0)|}\, \tilde{y}.$$

Applied to (10), this gives

$$\tilde{\xi} = -(1+\beta)\xi + \operatorname{sign}(H(0))\, \xi^3 = -(1+\beta)\xi \pm \xi^3,$$

which is the normal form (9).

□

2.4 Q4: Logistic map $x \mapsto \alpha x(1-x)$

We now analyse the period-doubling bifurcation(s) of the logistic family

$$F_\alpha(x) = \alpha x(1-x), \qquad x \in \mathbb{R},\ \alpha \in \mathbb{R},$$

We first restrict the parameter range and phase space to the invariant region where the dynamics take place. For $\alpha \in [0,4]$ the interval $[0,1]$ is forward invariant, since for $x \in [0,1]$,

$$0 \le F_\alpha(x) = \alpha x(1-x) \le \frac{\alpha}{4} \le 1.$$

Hence we restrict to $\alpha \in [0,4]$ and $x \in [0,1]$.

Fixed points and their stability. Fixed points satisfy $F_\alpha(x) = x$, i.e.

$$x_0(\alpha) = 0, \qquad x_1(\alpha) = 1 - \frac{1}{\alpha} \quad (\alpha \neq 0).$$

The derivative is

$$F'_\alpha(x) = \alpha(1 - 2x).$$

Thus x_0 is attracting if $|F'_\alpha(0)| = |\alpha| < 1$ and repelling if $|\alpha| > 1$. For $\alpha > 1$ the nontrivial fixed point satisfies

$$F'_\alpha(x_1(\alpha)) = \alpha\Big(1 - 2\big(1 - \tfrac{1}{\alpha}\big)\Big) = 2 - \alpha,$$

so $x_1(\alpha)$ is attracting for $|2 - \alpha| < 1$, i.e. $1 < \alpha < 3$, and repelling for $\alpha > 3$. At $\alpha = 3$ we have

$$x_1(3) = \frac{2}{3}, \qquad F'_3(x_1(3)) = -1,$$

so the fixed point $x_1(\alpha)$ satisfies the flip multiplier condition at $(x^*, \alpha_0) = (2/3, 3)$.

Genericity at the first flip ($\alpha = 3$). After the translation $u = x - \frac{2}{3}$ and reparametrization $\beta = \alpha - 3$, the bifurcation point becomes $(u, \beta) = (0, 0)$, so Theorem 5.27 from Lecture 10 applies; equivalently, we may verify (B.1)–(B.2) by evaluating the derivatives of $f(x, \alpha)$ at the original point $(x_0, \alpha_0) = (\frac{2}{3}, 3)$.

We now verify the genericity conditions (B.1)–(B.2) at $\alpha = 3$ for$f(x, \alpha) := F_\alpha(x) = \alpha x(1 - x)$,, evaluated at the fixed point where the multiplier equals -1, i.e. $(x_0, \alpha_0) = (\frac{2}{3}, 3)$

The genericity conditions are:

$$\text{(B.1)} \quad \frac{1}{2}\big(f_{xx}(x_0, \alpha_0)\big)^2 + \frac{1}{3} f_{xxx}(x_0, \alpha_0) \neq 0, \qquad \text{(B.2)} \quad f_{x\alpha}(x_0, \alpha_0) + \frac{1}{2} f_\alpha(x_0, \alpha_0)\, f_{xx}(x_0, \alpha_0) \neq 0.$$

We compute the derivatives:

$$f_{xx}(x, \alpha) = -2\alpha, \qquad f_{xxx}(x, \alpha) = 0, \qquad f_\alpha(x, \alpha) = x(1 - x), \qquad f_{x\alpha}(x, \alpha) = 1 - 2x.$$

Hence

$$\text{(B.1)} \quad \frac{1}{2}\big(f_{xx}(x_0, \alpha_0)\big)^2 + \frac{1}{3} f_{xxx}(x_0, \alpha_0) = \frac{1}{2} \cdot 36 + 0 = 18 \neq 0,$$

and

$$\text{(B.2)} \quad f_{x\alpha}(x_0, \alpha_0) + \frac{1}{2} f_\alpha(x_0, \alpha_0)\, f_{xx}(x_0, \alpha_0) = -\frac{1}{3} + \frac{1}{2}\Big(\frac{2}{9}\Big)(-6) = -1 \neq 0.$$

Thus the flip at $\alpha = 3$ is generic.

Type of the flip at $\alpha = 3$. At $\alpha_0 = 3$ we have $f_{xx}(x, \alpha) = -2\alpha$ and $f_{xxx}(x, \alpha) = 0$, hence

$$b(\alpha_0) = \tfrac{1}{2} f_{xx}(x_0, \alpha_0) = -3, \qquad c(\alpha_0) = \tfrac{1}{6} f_{xxx}(x_0, \alpha_0) = 0,$$

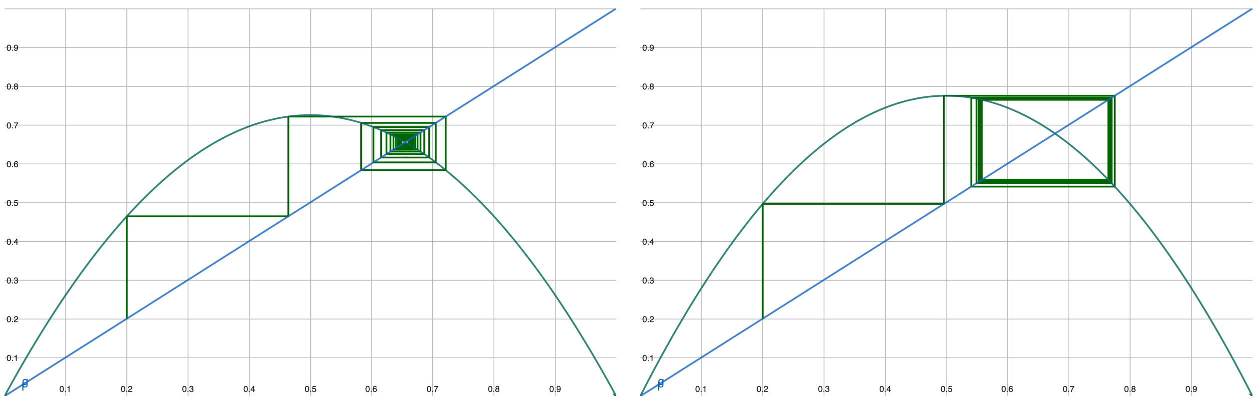
so the cubic normal form coefficient after elimination of the quadratic term is

$$H(\alpha_0) = c(\alpha_0) + b(\alpha_0)^2 = 9 > 0.$$

Along the fixed point branch $x_1(\alpha)$ we have $\mu(\alpha) = 2 - \alpha$, hence $\mu'(\alpha_0) = -1 \neq 0$. Thus we may take the unfolding parameter $\beta = \alpha - 3$.

Therefore the flip at $\alpha = 3$ is *supercritical*: for $\alpha > 3$ a unique attracting period-2 orbit is created, while the fixed point $x_1(\alpha)$ becomes repelling.

Figure 2 illustrates the orbit picture of a flip bifurcation using cobweb diagrams for the logistic map. The graph of F_α is plotted together with the diagonal $y = x$, and the staircase shows successive iterates starting from the same initial value x_0. For $\alpha = 2.9 < 3$ the staircase converges to the stable fixed point, while for $\alpha = 3.1 > 3$ the fixed point has become unstable and the iterates approach a stable period-2 orbit, i.e. they alternate between two distinct values.



(a) $\alpha = 2.9$ (convergence to a stable fixed point).

(b) $\alpha = 3.1$ (convergence to a period-2 orbit).

Figure 2: Cobweb diagrams for the logistic map $F_\alpha(x) = \alpha x(1 - x)$ illustrating the flip at $\alpha = 3$.

The period-2 orbit and its next flip. A point has period 2 (and is not a fixed point) iff it is a non-fixed solution of

$$F_\alpha^2(x) = x.$$

We compute that

$$F_\alpha^2(x) - x = (F_\alpha(x) - x)\, Q_\alpha(x),$$

where $F_\alpha(x) - x = 0$ gives the fixed points and Q_α is a quadratic polynomial. The two roots of Q_α are

$$x_\pm(\alpha) = \frac{\alpha + 1 \pm \sqrt{\alpha^2 - 2\alpha - 3}}{2\alpha} = \frac{\alpha + 1 \pm \sqrt{(\alpha - 3)(\alpha + 1)}}{2\alpha}.$$

For $\alpha > 3$ the discriminant is positive, so $x_+(\alpha) \neq x_-(\alpha)$ and these two points form a period-2 orbit. (At $\alpha = 3$ the two roots coalesce at $x = 2/3$.)

The multiplier of a period-2 orbit $\{x_+, x_-\}$ is by definition the product of the derivatives over one period. By the chain rule this equals the derivative of the second iterate at either point (since $F_\alpha(x_+) = x_-$ and $F_\alpha(x_-) = x_+$):

$$(F_\alpha^2)'(x_+) = F_\alpha'(F_\alpha(x_+))\, F_\alpha'(x_+) = F_\alpha'(x_-)\, F_\alpha'(x_+).$$

Using $F'_\alpha(x) = \alpha(1-2x)$ and the identities

$$x_+ + x_- = \frac{\alpha+1}{\alpha}, \qquad x_+x_- = \frac{\alpha+1}{\alpha^2},$$

we obtain

$$F'_\alpha(x_+)F'_\alpha(x_-) = \alpha^2(1-2x_+)(1-2x_-) = \alpha^2\big(1-2(x_++x_-)+4x_+x_-\big) = -\alpha^2+2\alpha+4.$$

Denote $m(\alpha) := (F^2_\alpha)'(x_\pm(\alpha)) = -\alpha^2+2\alpha+4$.

To visualize the stability, we plot $m(\alpha)$ together with the reference lines $m=1$ and $m=-1$.

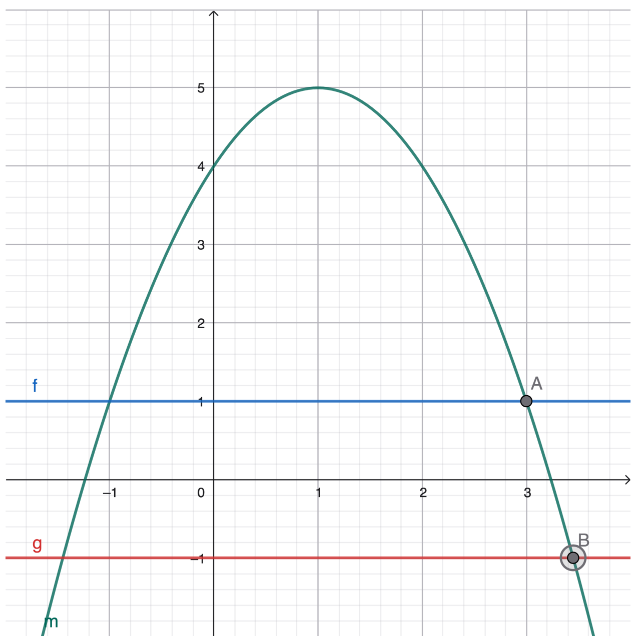


Figure 3: Multiplier of the period-2 orbit in the logistic map: $m(\alpha) = (F^2_\alpha)'(x_\pm(\alpha)) = -\alpha^2+2\alpha+4$. The intersections with $m=1$ and $m=-1$ occur at $\alpha=3$ and $\alpha_2 = 1+\sqrt{6}$, respectively.

As shown in Figure 3, we have $m(3)=1$, consistent with the fact that the period-2 orbit is created at $\alpha=3$. Moreover, the next flip of the period-2 orbit occurs when $m(\alpha)=-1$, which happens at

$$-\alpha^2+2\alpha+4=-1 \iff \alpha^2-2\alpha-5=0 \iff \alpha_2 = 1+\sqrt{6}.$$

Therefore the period-2 orbit is attracting for $3<\alpha<1+\sqrt{6}$, and at $\alpha=\alpha_2$ it satisfies the flip condition as a fixed point of F^2_α (its multiplier equals -1).

3 Conclusion

In this essay we studied period-doubling bifurcations in one-dimensional maps. First, we described the basic setup where a fixed point loses stability when the multiplier crosses -1. Then we proved that a unique nearby 2-cycle is created and we determined its stability. We then showed that, after a smooth change of coordinates and a rescaling, the dynamics near the bifurcation are locally topologically equivalent to the normal form

$$\xi \mapsto -(1+\beta)\xi \pm \xi^3.$$

Finally, we applied the theory to the logistic map $F_\alpha(x) = \alpha x(1-x)$. We found the first generic flip at $\alpha = 3$, where the fixed point becomes unstable and a stable period-2 orbit is born.

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