Nuclear Physics B256 (1985) 727-745

ON THE QUANTUM STRUCTURE OF A BLACK HOLE

Gerard 't HOOFT

Institute for Theoretical Physics, Princetonplein 5, PO Box 80.006, 3508 TA Utrecht, The Netherlands

Received 26 October 1984
(Revised 30 November 1984)

The assumption is made that black holes should be subject to the same rules of quantum mechanics as ordinary elementary particles or composite systems. Although a complete theory for reconciling this requirement with that of general coordinate transformation invariance is not yet in sight, a number of observations can be made and a general framework is suggested.

1. Introduction

In view of the fundamental nature of both the theory of general relativity and that of quantum mechanics there seems to be little need to justify any attempt to reconcile the two theories. Yet the essential academic interest of the problem may be not our only motive. It seems to become more and more clear that "ordinary" elementary particle physics, in energy regions that will be accessible to machines in the near future, is plagued by mysteries that may require a more drastic approach than usually considered: the so-called hierarchy problems, and the freedom to choose coupling constants and masses. A variant of these problems also occurs in quantum gravity. Here it is the mystery of the vanishing cosmological constant. It would not be the first time in history if a solution to these mysteries could be found by first contemplating gravity theory; after all, the very notion of Yang-Mills fields was inspired by general relativity. In this paper we will present an approach in which the cosmological constant problem is acute and so, any progress made might help us in particle physics also.

We are not yet that far. The aim of the present paper is to set the scene, to provide a new battery of formulas that may become useful one day.

To many practitioners of quantum gravity the black hole plays the role of a soliton, a non-perturbative field configuration that is added to the spectrum of particle-like objects only after the basic equations of their theory have been put down, much like what is done in gauge theories of elementary particles, where Yang-Mills equations with small coupling constants determine the small-distance structure, and solitons and instantons govern the large-distance behavior.

Such an attitude however is probably not correct in quantum gravity. The coupling constant increases with decreasing distance scale which implies that the smaller the distance scale, the stronger the influences of "solitons". At the Planck scale it may well be impossible to disentangle black holes from elementary particles. There simply is no fundamental difference. Both carry a finite Schwarzschild radius and both show certain types of interactions. It is natural to assume that at the Planck length these objects merge and that the same set of physical laws should cover all of them.

Now in spite of the fact that the properties of larger black holes appear to be determined by well-known laws of physics there are some tantalizing paradoxes as we will explain further. Understanding these problems may well be crucial before one can proceed to the Planck scale.

At present a black hole is only (more or less) understood as long as it is in a quantum mechanically mixed state. The standard picture is that of the Hawking school [1, 3], but a competing description exists [2] which this author was unable to rule out entirely, and which predicts a different radiation temperature. The question which of these pictures is right will not be considered in this paper; we leave it open by admitting a free parameter λ in the first sections.

Whatever the value of λ, the picture is incomplete. If black holes show any resemblance with ordinary particles it should be possible to describe them as *pure* states, even while they are being born from an implosion of ordinary matter, or while the opposite process, evaporation by Hawking radiation, takes place. As we will argue, any attempt to "unmix" the black-hole configuration (that is, produce a density matrix with eigenvalues closer to 1 and 0) produces matter in the view of a freely falling observer. It is this "matter" that we will be considering in this paper.

We start with the postulate that there exists an extension of Hilbert space comprising black holes, and that a hamiltonian can be precisely defined in this Hilbert space, although a certain amount of ambiguity due to scheme (gauge) dependence is to be expected. Although at first sight this postulate may seem to be nearly empty, it is directly opposed to conclusions of the Hawking school [1]. These authors apply the rule of invariance under general coordinate transformations in the conventional way. We believe that, although our present dogma may well prove wrong ultimately, it stands a good chance of being correct if we apply general coordinate transformations more delicately, and in particular if the distinction between "vacuum" and "matter" may be assumed to be observer dependent when coordinate transformations with a horizon are considered.

Another point where the conventional derivations could be greeted with some scepticism is the role of the infalling observer. If the infalling observer sees a pure state an outside observer sees a mixed state. In our view this could be due simply to the fact that the "infalling observer" makes part of the system seen by the outsider: he himself is included in the outside Hilbert space. Clearly the very foundations of quantum mechanics are touched here and we leave continuation of this intriguing subject to philosophers, rather than accepting the simple-minded conclusion that transitions will take place between pure states and mixed states [1].

Nowhere a distinction may be made between "primordial" black holes and black holes that have been formed by collapse. Now we will make use of space-time metrics that contain a future and a past horizon, and at first sight this seems to be a misrepresentation of the history of a black hole with a collapse in its past. However, just because no distinction is made one is free to choose whichever metric is most suitable for a description of the present state of a black hole. It may be expected that in a pure-state description of a black hole a collapsing past will be very difficult to describe if this collapse took place much longer ago than at time $t = -M \log M$ in Planck units.

From now on a black hole will be defined to be any particle which is considerably heavier than the Planck mass and which shows only the minimal amount of structure (nothing much besides its Hawking radiation [3]) outside its Schwarzschild horizon.

The first part of this paper gives a number of pedestrian arguments indicating what kind of quantum structure one might expect in a black hole. Although they should not be considered as airtight mathematical proofs this author finds it extremely hard to imagine how the conclusions of sect. 2 of this paper could be avoided:

(i) The spectrum of black holes states is discrete. The density of states for heavy black holes can be computed up to an (unknown but finite) overall constant.

(ii) Baryon number conservation, like all other additive quantum numbers to which no local gauge field is coupled, must be violated.

The first of these conclusions is related to the expected thermodynamic properties first proposed by Bekenstein [4]. The second has been discussed at length by him and also by Zeldovich [5].

In sect. 3 a naive model ("brick wall model") is constructed that roughly reproduces these features. Being a model rather than a theory it violates the fundamental requirement of coordinate invariance at the horizon so it cannot represent a satisfactory solution to our problem, but in its simplicity it does show some important facts: it is the horizon itself, rather than the black hole as a whole that determines its quantum properties.

In the second part of the paper we formulate a much more precise and satisfying approach. A set of coordinate frames and transformation laws is proposed. One simplification is made that presumably is fairly harmless: we assume that, averaged over a certain amount of time, ingoing and outgoing particles near a black hole are smeared equally over all angles θ, φ. Certainly this is true for a Hawking-radiating Schwarzschild black hole. Furthermore, ingoing things may be *chosen* to be spherically smeared. As a result we may take the gravitational interactions between ingoing and outgoing matter (our most crucial problem) to be rotationally invariant. This assumption appears to be quite suitable for a first attempt to obtain an improved theory, and in fact it makes our whole approach quite powerful.

As stated before, how to interpret the coordinate transformations physically is yet another matter. The unorthodox interpretation that we proposed in ref. [2] is not ruled out (in fact it fits quite well in our description) but we do not insist on it. Rather, we allow the reader to draw his own conclusions here.

It is important to note that never space-times are considered that contain a "conical singularity" at the origin or elsewhere. Even in the formalism with $\lambda = 2$ there is no such singularity. This is because the "identification" of the points x with the points $-x$ is Minkowski space of ref. [2] only is made in the *classical limit*, not in the quantum theory.

Quantization of the space-time metric $g_{\mu\nu}$ itself is not considered explicitly, since we attempt to deduce the black-hole's properties from known laws of physics much beyond the Planck length. This may be incorrect: it could be that *only* by considering the full Hilbert space of all metrics a workable picture emerges. Our attitude is to first keep everything as simple as possible and only accept such complications when they clearly become unavoidable.

2. The black hole spectrum

A black hole can absorb particles according to the well-known laws of general relativity: the geodesics of particles with an impact parameter below a certain threshold will disappear into the horizon.

Conversely, black holes may emit particles as derived by Hawking. They radiate as black bodies with a certain temperature:

$$T_{\mathrm{H}} = \lambda / 8\pi M, \tag{2.1}$$

in units where the gravitational constant, the speed of light, Planck's constant and Boltzmann's constant are put equal to one. As is explained in sect. 1 there may be reasons to put into question the usual argument that $\lambda = 1$. There is an alternative theory with $\lambda = 2$, but the precise value of λ is of little relevance to the following argument.

If a black hole is to be compared with any ordinary quantum mechanical system such as a heavy atomic nucleus or any "black box" containing a number of particles and possessing a certain set of energy levels, which furthermore can absorb and emit particles in a similar fashion, then a conclusion on the density of its energy levels, $\rho(E)$, can readily be drawn.

Imagine an object with energy ΔE being dropped into a black hole with mass E, so that the final mass is $E + \Delta E$, where in Planck units

$$\Delta E \ll 1 \ll E. \tag{2.2}$$

Let R be the bound on the impact parameter; usually

$$R \simeq 2E. \tag{2.3}$$

Then the absorption cross section σ is

$$\sigma = \pi R^2. \tag{2.4}$$

From Hawking's result we conclude that the emission probability W is

$$W \simeq \pi R^2 \rho_{\Delta E}\, e^{-\beta_H \Delta E}\,, \qquad (2.5)$$

where $\rho_{\Delta E}$ is the density of states for a particle with energy ΔE per volume element, and β_H is the inverse of the temperature T_H.

Now if the same processes can be described by a hamiltonian acting in Hilbert space, then we should have in the first case

$$\sigma = |\langle E+\Delta E|T|E, \Delta E\rangle|^2 \rho(E+\Delta E)\,, \qquad (2.6)$$

where T is the scattering matrix, and in the second case

$$W = |\langle E, \Delta E|T|E+\Delta E\rangle|^2 \rho(E)\rho_{\Delta E}\,, \qquad (2.7)$$

by virtue of the "golden rule".

The matrix elements in both cases should be equal to each other if PCT invariance is to be respected (the expressions hold for particles and antiparticles equally). So dividing the two expressions we find

$$\frac{\rho(E+\Delta E)}{\rho(E)} = e^{+\beta_H \Delta E}\,, \qquad (2.8)$$

$$\rho(E) = C\, e^{4\pi\lambda^{-1}E^2}\,. \qquad (2.9)$$

Now this result could also have been concluded from the usual thermodynamical arguments [4]. The entropy S is

$$S = 4\pi\lambda^{-1}E^2 + \text{const}\,, \qquad (2.10)$$

from which indeed (2.9) follows. We note that E^2 in (2.9) and (2.10) measures the total area of the horizon.

The constant in (2.9) and (2.10) is not known. Could it be infinite? We claim that this can only be the case if there exists a "lightest" *stable* black hole. Just compare eqs. (2.6) and (2.7) if $E+\Delta E$ represents the lightest black hole, and E and ΔE are ordinary elementary particles. If $\rho(E+\Delta E)$ is infinite but $\rho(E)$ finite then, since σ must be finite, W vanishes. Since this object fails to obey the classical laws of physics (it ought to emit Hawking radiation), it cannot be much heavier than the Planck mass. Now it is very difficult to conceive of any quantum theory that can admit the presence of such infinitely degenerate "particles". They are coupled to the graviton in the usual way, so their contributions to graviton scattering amplitudes and propagators would diverge with the constant C, basically because the probability for pair creation of this object in any channel with total energy exceeding the mass threshold is proportional to C.

Clearly the above arguments assume that several familiar concepts from particle field theory such as unitarity, causality, positivity, etc. also apply to these extreme energy end length scales, and one might object against making such assumptions. But we considered this to be a reasonable starting point and henceforth assume C

to be finite. An interesting but as yet not much explored possibility is that C, though finite, might be extremely large. After all, large numbers such as $M_{\text{Planck}}/m_{\text{proton}}$ are unavoidable in this area of physics. One might find interesting links with the $1/N$ expansion suggested by Weinberg [6], who however finds physically unacceptable poles in the $N \to \infty$ limit.

It is rather unlikely that C is exactly constant. One expects subdominant terms in the exponent. Also, we have not taken into account the other degrees of freedom such as angular momentum and electric (possibly also magnetic) charge. These can be taken into account by rather straightforward extrapolations.

Other additive quantum numbers cannot possibly be conserved. The reason is that the larger black hole may absorb baryons or any other such objects in unlimited quantities. If indeed it allows only a finite number of quantum states then any assignment of baryon number will fail sooner or later. Indeed one of the paradoxes we will have to face is that starting with a theory that is invariant under rotations in baryon number space one must end up with a theory where this invariance is broken [5].

3. The brick wall model

When one considers the number of energy levels a particle can occupy in the vicinity of a black hole one finds a rather alarming divergence at the horizon. Indeed, the usual claim that a black hole is an infinite sink of information (and a source of an ideally random black body radiation of particles [3, 7]) can be traced back to this infinity. It is inherent to the arguments in the previous sections that this infinity is physically unacceptable. Later in this section we will see that, as is often the case with such infinities, the classical treatment of the infinite parts of these expressions is physically incorrect. The particle wave functions extremely close to the horizon must be modified in a complicated way by gravitational interactions between ingoing and outgoing particles (sects. 5 and 6). Before attempting to consider these interactions properly we investigate the consequences of a simple-minded cut-off in this section.

It turns out to be a good exercise to see what happens if we assume that the wave functions must all vanish within some fixed distance h from the horizon:

$$\varphi(x)=0 \quad \text{if} \quad x \leq 2M+h\,, \tag{3.1}$$

where M is the black hole mass. For simplicity we take $\varphi(x)$ to be a scalar wave function for a light ($m \ll 1 \ll M$) spinless particle. Later we will give them a multiplicity Z as a first attempt to mimic more closely the real world.

In the view of a freely falling observer, condition (3.1) corresponds to a uniformly accelerated mirror which in fact will create its own energy-momentum tensor due to excitation of the vacuum. As in sect. 1 we stress that this presence of matter and energy may be observer dependent, but above all this model should be seen as an elementary exercise, rather than an attempt to describe physical black holes accurately.

Let the metric of a Schwarzschild black hole be given by

$$\mathrm{d}s^2=-\left(1-\frac{2M}{r}\right)\mathrm{d}t^2+\left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2+r^2\,\mathrm{d}\Omega^2\,. \tag{3.2}$$

Furthermore, we need an "infrared cutoff" in the form of a large box with radius L:

$$\varphi(x)=0 \quad \text{if} \quad x=L\,. \tag{3.3}$$

The quantum numbers are l, l_3 and n, standing for total angular momentum, its z-component and the radial excitations. The energy levels $E(l, l_3, n)$ can then be found from the wave equation

$$\left(1-\frac{2M}{r}\right)^{-1}E^2\varphi+\frac{1}{r^2}\partial_r r(r-2M)\partial_r\varphi-\left(\frac{l(l+1)}{r^2}+m^2\right)\varphi=0\,. \tag{3.4}$$

As long as $M \gg 1$ (in Planck units) we can rely on a WKB approximation. Defining a radial wave number $k(r, l, E)$ by

$$k^2=\frac{r^2}{r(r-2M)}\left(\left(1-\frac{2M}{r}\right)^{-1}E^2-r^{-2}l(l+1)-m^2\right), \tag{3.5}$$

as long as the r.h.s. is non-negative, and $k^2=0$ otherwise, the number of radial modes n is given by

$$\pi n=\int_{2M+h}^{L}\mathrm{d}r\,k(r, l, E)\,. \tag{3.6}$$

The total number N of wave solutions with energy not exceeding E is then given by

$$\begin{aligned}\pi N&=\int(2l+1)\,\mathrm{d}l\,\pi n \overset{\text{def}}{=} g(E)\\ &=\int_{2M+h}^{L}\mathrm{d}r\left(1-\frac{2M}{r}\right)^{-1}\int(2l+1)\,\mathrm{d}l\sqrt{E^2-\left(1-\frac{2M}{r}\right)\left(m^2+\frac{l(l+1)}{r^2}\right)}\,,\end{aligned} \tag{3.7}$$

where the l-integration goes over those values of l for which the argument of the square root is positive.

What we have counted in (3.7) is the number of classical eigenmodes of a scalar field in the vicinity of a black hole. We now wish to find the thermodynamic properties of this system such as specific heat etc. Every wave solution may be occupied by any integer number of quanta. Thus we get for the free energy F at some inverse temperature β,

$$\mathrm{e}^{-\beta F}=\sum \mathrm{e}^{-\beta E}=\prod_{n,l,l_3}\frac{1}{1-\mathrm{e}^{-\beta E}}\,, \tag{3.8}$$

or

$$\beta F=\sum_{N}\log\left(1-e^{-\beta E}\right); \tag{3.9}$$

and, using (3.7),

$$\begin{aligned}\pi\beta F&=\int dg(E)\log\left(1-e^{-\beta E}\right)\\&=-\int_0^\infty dE\frac{\beta g(E)}{e^{\beta E}-1}\\&=-\beta\int_0^\infty dE\int_{2M+h}^{L}dr\left(1-\frac{2M}{r}\right)^{-1}\int\left(2l+1\right)dl\\&\quad\times(e^{\beta E}-1)^{-1}\sqrt{E^2-\left(1-\frac{2M}{r}\right)\left(m^2+\frac{l(l+1)}{r^2}\right)}.\end{aligned} \tag{3.10}$$

Again the integral is taken only over those values for which the square root exists. In the approximation

$$m^2\ll 2M/\beta^2h,\qquad L\gg 2M, \tag{3.11}$$

we find that the main contributions are

$$F\simeq-\frac{2\pi^3}{45h}\left(\frac{2M}{\beta}\right)^4-\frac{2}{9\pi}L^3\int_m^\infty\frac{dE(E^2-m^2)^{3/2}}{e^{\beta E}-1}. \tag{3.12}$$

The second part is the usual contribution from the vacuum surrounding the system at large distances and is of little relevance here. The first part is an intrinsic contribution from the horizon and it is seen to diverge linearly as $h\to 0$.

The contribution of the horizon to the total energy U and the entropy S are

$$U=\frac{\partial}{\partial\beta}(\beta F)=\frac{2\pi^3}{15h}\left(\frac{2M}{\beta}\right)^4Z, \tag{3.13}$$

$$S=\beta(U-F)=\frac{8\pi^3}{45h}2M\left(\frac{2M}{\beta}\right)^3Z. \tag{3.14}$$

We added a factor Z denoting the total number of particle types.

Let us now adjust the parameters of our model such that the total entropy is

$$S=4\lambda^{-1}M^2, \tag{3.15}$$

as in eq. (2.10), and the inverse temperature is

$$\beta=8\pi\lambda^{-1}M. \tag{3.16}$$

This is seen to correspond to

$$h=\frac{Z\lambda^4}{720\pi M}. \tag{3.17}$$

Note also that the total energy is

$$U=\tfrac{3}{8}M, \tag{3.18}$$

independent of Z, and indeed a sizeable fraction of the total mass M of the black hole! We see that it does not make much sense to let h decrease much below the critical value (3.17) because then more than the black hole mass would be concentrated *at our side* of the horizon.

Eq. (3.17) suggests that the distance of the "brick wall" from the horizon depends on M, but this is merely a coordinate artifact. The invariant distance is

$$\int_{r=2M}^{r=2M+h} \mathrm{d}s = \int \frac{\mathrm{d}r}{\sqrt{1-2M/r}} = 2\sqrt{2Mh} = \sqrt{\frac{Z\lambda^4}{90\pi}}. \tag{3.19}$$

Thus, the brick wall may be seen as a property of the horizon independent of the size of the black hole.

The conclusion of this section is that not only the infinity of the modes near the horizon should be cut-off, but also the value for the cut-off parameter is determined by nature, and a property of the horizon only. The model described here should be a reasonable description of a black hole as long as the particles near the horizon are kept at a temperature as given by (3.16) and all chemical potentials are kept close to zero. The reader is invited to investigate further properties of the model such as the average time spent by one particle near the horizon, etc.

The model automatically preserves quantum coherence completely, but it is also unsatisfactory: there might be several conserved quantum numbers, such as baryon number*. What is wrong, clearly, is that we abandoned the principle of invariance under coordinate transformations at the horizon. The question that we should really address is how to keep not only the quantum coherence but also general invariance, while dropping all global conservation laws.

4. Kruskal coordinates and the generators of time translation

The global structure of the Schwarzschild metric is more conveniently expressed in the well-known Kruskal coordinates u and v:

$$uv=\left(1-\frac{r}{2M}\right)\mathrm{e}^{r/2M}, \tag{4.1a}$$

$$v/u=-\mathrm{e}^{t/2M}. \tag{4.1b}$$

The metric takes the form

$$\mathrm{d}s^2=\frac{-32M^3}{r}\mathrm{e}^{\;r/2M}\,\mathrm{d}u\,\mathrm{d}v+r^2\,\mathrm{d}\Omega^2, \tag{4.2}$$

which remains regular at $r=2M$.

* One may postpone this difficulty by inserting explicitly baryon number violating interactions near the horizon.

We now define two kinds of "universes" described essentially by the same metric, but differing in their notions of a time coordinate. One of these we will denote as the "Kruskal universe", in which $u+v$ serves as a time coordinate. The other is the "Schwarzschild universe", in which the coordinate t plays the role of time.

At $t=0$, the horizon $(r=2M)$ divides 3-space into two regions:

$$\text{I:} \qquad u<0,\, v>0\,, \tag{4.3}$$

$$\text{II:} \qquad u>0,\, v<0\,. \tag{4.4}$$

The metrics in regions I and II are identical (this statement remains true in the Kerr, Reissner-Nordstrom and Kerr-Newman solutions), so we can observe that the Schwarzschild world falls apart into two worlds between which communication is not possible. Thus, a general coordinate transformation transforms one single Kruskal world into these *two* identical Schwarzschild worlds. One state in the Hilbert space in the Kruskal coordinates is mapped into a product state $|\psi\rangle_{\rm I}|\psi\rangle_{\rm II}$ of the double Schwarzschild universe.

Consider now an infinitesimal time translation in the Schwarzschild universes:

$$\begin{aligned} t &\to t+\varepsilon\,, \\ v &\to (1+\varepsilon/4M)v\,, \\ u &\to (1-\varepsilon/4M)u\,. \end{aligned} \tag{4.5}$$

If a Kruskal "hamiltonian" $h=\int h(v)\,\mathrm{d}v$ transforms v and u as

$$\begin{aligned} v &\to v+\varepsilon\,, \\ u &\to u+\varepsilon\,, \end{aligned} \tag{4.6}$$

then at $t=0$ with $u=-v$ we have for the generator $\mathcal{H}$ that generates (4.5)

$$\mathcal{H}=\int \frac{v}{4M}\,h(v)\,\mathrm{d}v\,. \tag{4.7}$$

Since in space II time runs backwards the hamiltonian in the double Schwarzschild universe can be written as

$$\mathcal{H}=H_{\rm I}-H_{\rm II}\,, \tag{4.8}$$

simply by splitting the integral (4.7) into its right-hand side and left-hand side. There are now two possibilities:

(i) Space II is inaccessible to us. If some amount of information - a particle for instance - has wandered into space II it will be lost forever. We must average over all states in II and if for example gravitational interactions couple space II to space I then pure quantum states may turn into mixed states.

(ii) Space II also represents our space. In that case a most convenient interpretation of (4.5) is that $\mathcal{H}$ dictates the evolution of a density matrix in our (Schwarzschild) world. Again, if certain couplings occur between I and II, pure states may turn into mixed states. As explained in ref. [2] the two options (i) and (ii) are quite inequivalent. Option (ii) gives the new value $\lambda = 2$ in eq. (2.1). Rather than attempting to determine here which of the options represents reality more closely we will concentrate on the transformation (4.7). How is it affected when gravitational interactions are switched on and how can we then describe the Hilbert spaces involved.

5. Infalling or outgoing matter

Let us define the Kruskal vacuum, $|0\rangle_K$ as the lowest eigenstate of the operator h in Hilbert space. A Schwarzschild vacuum $|0\rangle_S$ is then the lowest eigenstate of H_I. As is well known, $|0\rangle_K$ is quite different from the product of two $|0\rangle_S$ states. Let us write the states in space II as bra states, $\langle\psi|$, in order to incorporate the minus sign in (4.8). Then the Schwarzschild vacuum

$$|0\rangle_S \cdot {}_S\langle 0| ,$$

where ${}_S\langle 0|$ is the lowest eigenstate of H_{II}, is a superposition of eigenstates of h with quite high eigenvalues: rather than a vacuum we have a tremendously energetic collision of particles in Kruskal space, centered about $u = v = 0$. In fact, any pure state in Schwarzschild space, whether it be one of the form $|\psi\rangle\langle\psi|$ or $|\psi\rangle_S\langle 0|$, corresponds to such an infinitely (?) energetic collision process in Kruskal space. Therefore, if we wish to describe pure states in Schwarzschild space, we must take gravitational interactions in Kruskal space into account. Now here is our problem. Superficially it seems that we have infinite energy in Kruskal space. What are the consequences for the metric?

We try to answer this question step by step. Let us first consider *infalling* particles only. It will be sufficient to consider only light particles ($m \ll 1$) whose energies were of the order of the Hawking temperature when they were still far from the horizon. Consider the process of "purifying" a state in Schwarzschild coordinates, starting from the Kruskal vacuum and removing more and more ingoing particles closer and closer to the horizon. In the process we unavoidably *add* more and more particles in the Kruskal world. As we approach the horizon we are affecting particle states with increasing velocities in Kruskal space. Finally we are considering particles in Kruskal space whose energies are so huge that their gravitational fields are no longer negligible. They move along the u-axis ($v = 0$) and the gravitational fields of such particles were described in ref. [8]. See figs. 1, 2. The result is remarkably simple: the spaces I and II are just shifted along the u-axis by an amount $\delta u(\theta, \varphi)$. From ref. [8] one may deduce, by contour integration,

$$\delta u = Cp \int_{\theta}^{2\pi-\theta} \mathrm{d}z (\cos\theta - \cos z)^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}z\sqrt{3}} \tag{5.1}$$

if a particle with Kruskal momentum p went in at the north pole. C is a numerical

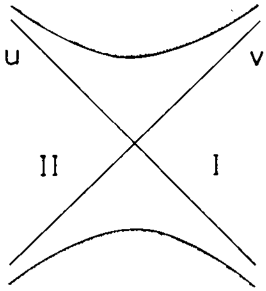

Fig. 1. The Kruskal vacuum. The u- and v-coordinates are shown. Shaded line is the $r = 0$ singularity.

constant:

$$C = 2^{9/2} M^4 \, e^{-1} (1 + e^{-\pi\sqrt{3}})^{-1} . \tag{5.2}$$

From (5.1) it is easy to see that δu is positive for all θ, φ.

If many particles fall in from different directions we find an average shift $\langle \delta u \rangle$, by integrating (5.1) over θ and φ:

$$\langle \delta u \rangle = 2^8 M^4 \, e^{-1} \langle p \rangle . \tag{5.3}$$

Now as we see from the transformation (4.1), a time translation over an interval Δt in Schwarzschild space corresponds to multiplying v and dividing u by an amount $e^{\Delta t/4M}$. In the new coordinates then p grows exponentially with Δt.

It now seems that the number of allowed modes for the other, soft particles in Kruskal coordinates increases with δu (see fig. 2), because more and more particles are allowed that pass the u-axis through the newly opened interval.

At first sight this situation does not improve if we also allow for *outgoing* matter. If we now proceed to remove the outgoing particles in Schwarzschild space, thus adding more and more outgoing particles in Kruskal space, then we may reach the

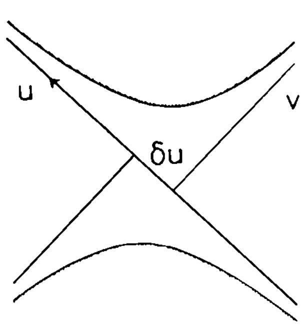

Fig. 2. A particle that may have been as light as a thermal proton went into the system some time (t) ago, giving a shift δu in the horizon increasing exponentially with time. δu also depends on the angles θ, φ diverging logarithmically where the proton hits the horizon.

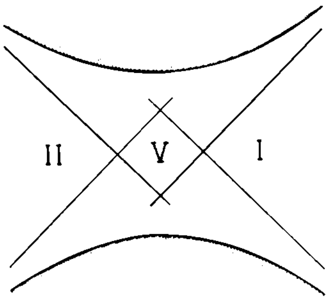

Fig. 3. The metric in Kruskal-like coordinates when incoming and outgoing matter are taken into account.

point where also their gravitational effects may become important. As long as their momenta are not too large we may assume that they superimpose on the effects of the incoming particles and we expect that the metric in Kruskal coordinates resembles fig. 3, where the future and past horizons are now shifted with respect to each other.

The most important feature that we notice is that a new space, labeled V in fig. 3, opens up. This seems to imply that as we attempt to remove the particles seen by a Schwarzschild observer when he looks at a Kruskal vacuum, this Kruskal space becomes larger, allowing for new particle modes living in space V.

We must stress that using eq. (5.1) now to express both the shifts in the u- and the v-coordinates is not exactly correct as soon as the product $\delta u\, \delta v$ becomes non-negligible. Fig. 3 is just a first approximation.

6. Spherical shells of matter

The exact metric when many particles are coming in and going out is expected to be cumbersome to compute, if this is possible at all. Nevertheless we wish to have some idea on what happens if both δu and δv become large. One might fear that space V grows indefinitely, which would jeopardize our attempts to formulate an equivalence principle for pure quantum states, because then a continuous spectrum of states in V would have to be considered.

It is here that another kind of approximation may be helpful: let us take the case that there are many ingoing and outgoing particles whose distribution is smeared equally over all angles θ and φ. Then spherical symmetry can be called upon and life will be simple again. To be precise, we now propose the following model. In Kruskal space particles are considered moving to and fro, and their gravitational fields are not neglected. However, in order to simplify calculations, we take the gravitational fields as if the particles were smeared over all angles θ and φ. It is likely then that the "back-reaction" of matter in this model may reproduce correctly the decrease or increase of the black hole mass (not yet its angular momentum), when the Hawking effect is considered. Furthermore, as we shall see, these spherically

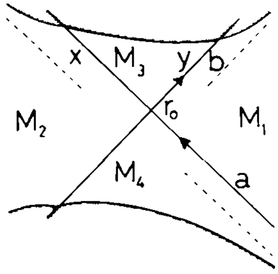

Fig. 4. A shell of matter a, is entering and b is leaving. The metric is then a patch of four Schwarzschild solutions with masses $M_1, \ldots, M_4$. The singularities at $r=0$ may or may not be hit. Note that the solid lines are here worldlines of matter, not horizons.

symmetric metrics are easy to compute and a quite remarkable pattern emerges. It is not unreasonable to assume that when in a more advanced version of this model non-spherically symmetric metrics are considered most of the topological features will remain the same.

The class of solutions that we will now consider has been derived in ref. [9]. The general spherically symmetric solution for two shells of matter going into and out of a black hole is pictured in fig. 4. Four, in general different, Schwarzschild solutions are matched at lightlike boundary lines (u or v are constant there), and the matching is defined by requiring the Schwarzschild coordinate r to be continuous on these lines. The rest of the metric could in principle be discontinuous but it is advisable to choose coordinates such that we have complete continuity, so that any possible conical singularity at r_0 (where the two shells meet) is avoided.

Let x and y be the two light-cone coordinates in the pattern of fig. 4. Let the matter worldlines be at $x=0$ and at $y=0$, which need not be horizons. In the regions $i(=1, \ldots, 4)$ we have the Kruskal coordinates

$$u_i = u_i(x)\,, \qquad v_i = v_i(y)\,, \tag{6.1}$$

and define

$$u_i' = \frac{\mathrm{d}u_i}{\mathrm{d}x}\,, \qquad v_i' = \frac{\mathrm{d}v_i}{\mathrm{d}y}\,. \tag{6.2}$$

In terms of the u_i and v_i the metric takes the form of eq. (4.2) with mass M_i, in the regions indicated in fig. 4. If we write this as

$$\mathrm{d}s^2 = -2A(x,y)\,\mathrm{d}x\,\mathrm{d}y + r^2(x,y)\,\mathrm{d}\Omega^2\,, \tag{6.3}$$

then at the line $x=0$ the quantity A is continuous if

$$\frac{u_3(0)}{M_3 u_3'(0)} = \frac{u_1(0)}{M_1 u_1'(0)} \stackrel{\text{def}}{=} \gamma_{13}\,, \tag{6.4}$$

and we have a similar condition at $y=0$. The momentum of the shell of matter causing the discontinuity can be calculated to be

$$K\frac{M_1 - M_3}{\gamma_{13}r^2} = \frac{p}{4\pi r^2}, \tag{6.5}$$

where K is a numerical constant (compare (5.3)):

$$K = 2^6 M_1^4 (e\pi)^{-1}, \tag{6.6}$$

if the momentum p is measured in terms of the coordinates in space I (the rather strange dimensionality of K is a coordinate artifact).

Consistency of the conditions (6.4) at the origin gives us

$$(r_0 - 2M_1)(r_0 - 2M_2) = (r_0 - 2M_3)(r_0 - 2M_4), \tag{6.7}$$

where r_0 is the radius of the shells when they meet at the origin. We find momentum conservation:

$$\frac{M_1 - M_3}{\gamma_{13}} = \frac{M_4 - M_2}{\gamma_{24}}. \tag{6.8}$$

It is important to require p in (6.5) to be always positive, so that the two sides of eq. (6.8) are also required to be positive. As a result we find the following general rules:

(i) When a particle line hits an $r = 0$ singularity the mass of the inner solution is always larger than the mass of the outer solution. For example, in fig. 4,

$$M_3 > M_1;\ M_3 > M_2;\ M_4 > M_2.$$

(ii) When a particle line escapes to infinity this is the other way around (due to a sign change of $u_i(0)$ in eq. (6.4)). Thus in fig. 4,

$$M_4 < M_1.$$

(iii) When a particle line coincides with a horizon ($u(x=0)=0$) then it connects two equal masses, as in fig. 2.

Formally these statements can be extended to the case when negative Schwarzschild masses are considered. However then

(iv) a negative mass singularity at $r = 0$ is timelike and must connect the future singularity with the past singularity.

This latter point makes negative mass solutions unattractive to work with but perhaps they cannot completely be excluded.

Now we ask the reader to contemplate once more fig. 3. When the solid lines in there represent shells of matter then the volume of space V will indeed grow with increasing momentum. At some values of the momenta region V will hit the $r = 0$ singularities. When the momenta are increased still further we now can say what will happen: the inner mass, μ, must be smaller than the outer mass M. *The condition $\mu \geq 0$ is a constraint on the total momentum allowed.* Fig. 5 depicts the situation when $\mu \ll M$.

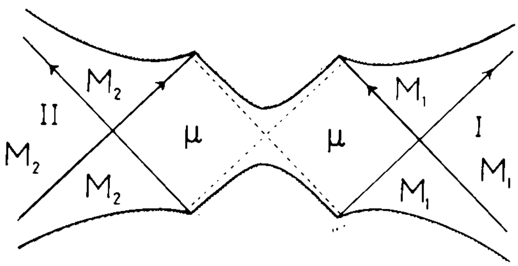

Fig. 5. Two sets of matter shells some in and out in spaces I and II. Inside we have a mass μ with $\mu < M_1$ and $\mu < M_2$.

We notice that as μ gets smaller, the spaces I and II become more and more disconnected. This is because the connection goes via the "tiny" wormhole of the mass-μ object in the center. Also, of course, we may have that $M_2 \neq M_1$, so indeed we are dealing with two universes I and II that now may contain different masses. In the language of ref. [2] one could say that here we have a density matrix built from a bra and a ket state representing different masses. A diagonal density matrix then has $M_1 = M_2$.

Most important must be the solution with $\mu = 0$, which was also described in ref. [9]. Now spaces I and II are entirely disconnected. From eqs. (5.3), (6.4), (6.5) and (6.8) we derive that this happens at

$$p_{\text{in}} \cdot p_{\text{out}} = (4\pi K)^2 M^2 \,. \tag{6.9}$$

Note that the arguments of sect. 5 would imply that both p_{in} and p_{out} tend to infinity in the process of purifying the states in Schwarzschild space. Now our condition (6.9) apparently puts this to a halt. And now we have the configuration depicted in fig. 6. It is tempting to propose a mapping between pure states in the space of fig. 6 and *pure* states in Schwarzschild space (see sect. 7).

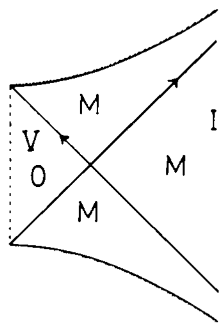

Fig. 6. The case $\mu = 0$. The dotted line is the coordinate singularity $r = 0$. There is one outer space, region I and a compact inner space, region V.

It is important to note that the shells of matter in fig. 6 are entering and leaving the black hole at the horizon. Since in eq. (6.9) p_{in} and p_{out} are finite this implies that for an observer in Schwarzschild space these amounts of matter are only infinitesimal. So one should *not* consider fig. 6 as the space-time picture of a black hole that is created by imploding matter and evaporated by the Hawking effect, terminating its life with some explosion. The total (Schwarzschild) time scale that we are at all considering is extremely short compared to a black hole's lifetime, rather it is of order $M \log M$ in Planck units.

7. Conclusion

If the eigenstates of an (either complete or unperturbed) hamiltonian are used to characterize a Hilbert space then clearly the choice of a coordinate frame affects this characterization. In particular if in a coordinate transformation the quantity

$$\partial t'/\partial t$$

does not everywhere have the same sign the two Hilbert spaces will look very different. In particular the notion of a vacuum will not at all coincide in the two systems and for that reason such transformations cannot be considered without taking into account the presence of matter. This is why we believe that in the usual Kruskal coordinates a black hole must be represented by more general metrics such as the ones discussed in the previous section. It now turns out that if too much matter is added in the Kruskal frame then the $r=0$ singularity in a certain region in the center becomes timelike because the metric there gets a Schwarzschild structure with negative mass parameter. In that case there will be no time slices (spacelike three-dimensional subspaces that intersect all timelike curves) without singularity. This presumably means that the amount of matter admitted in the Kruskal frame is limited. If the limit is saturated (eq. (6.9)) the Kruskal space-time becomes very special: there is *no* wormhole to another infinite universe; instead we do have a compact region, labeled V in fig. 6, not accessible for observers in Schwarzschild space.

We now have a proposal for the description of the pure quantum states of a black hole, which will partly replace the much uglier "brick wall model" of sect. 3. Postulating the absence of a wormhole we require a large but finite and precisely prescribed amount of matter in the Kruskal frame (eq. (6.9)). If we limit ourselves to regions close to the black hole (for instance by imposing a fairly harmless large distance limit L as in the "brick wall model"), Hilbert space will have only a finite number of states producing a mass, say, between M and $M+\delta M$. The evolution of these states as a function of Schwarzschild time can then be deduced from their evolution in the Kruskal frame.

Unfortunately there is a serious shortcoming, a defect, in this proposed model for a black hole: in the evolution of time p_{in} increases indefinitely while p_{out} decreases:

the model changes as a function of Schwarzschild time and therefore has no real predicting value. What is needed in addition is a mapping from states with most particles going in to states with more particles going out. A satisfactory prescription is not known as yet. The mapping $p_{in} \to p_{out}$ is like an (abelian) gauge transformation of which we do not yet know the transformation rules.

There may well be a good use for all those Kruskal systems which do have a wormhole (or possibly several) to another universe. In this case the amount of matter is less than the previous case, apart from a trivial doubling because of the two "universes". As we proposed previously [2], these systems could be considered as representing density matrix elements rather than pure states in Schwarzschild space. Now this proposal was greeted with considerable scepticism [10] which was a reason for the author not to use it as a backbone for this paper. Our general arguments are independent of such an assumption. But we do want to stress here that the density matrix interpretation fits quite naturally in our general picture. The "pure states" described in the beginning of this section do not follow a closed equation of motion in Schwarzschild space and therefore it is probably not directly possible to derive the Hawking temperature using them. The only system in our set of sect. 6 allowing for a computation of its evolution is the original Kruskal system without any matter present, because it is the only one with a timelike Killing vector. So it is not unreasonable to assume that this is the *only* system that allows us to compute the temperature (which in the density matrix interpretation comes out with twice the value, $\lambda = 2$, than the usual result, $\lambda = 1$), and that the pure state formalism of the beginning of this section is simply inadequate to derive λ at all (a naïve argument would suggest $\lambda = 1$, namely).

Let us finally remark that the density matrix interpretation of ref. [2] allows us to give a quite attractive interpretation of multiply connected space-times. The metric of two or more black holes, in the corresponding classical "Kruskal" frames, will show an equal number of wormholes. Because these wormholes always connect universes that are each other's twin image one may well decide to let every wormhole connect the *same pair* of universes (fig. 7). This is what *must* be done in our density

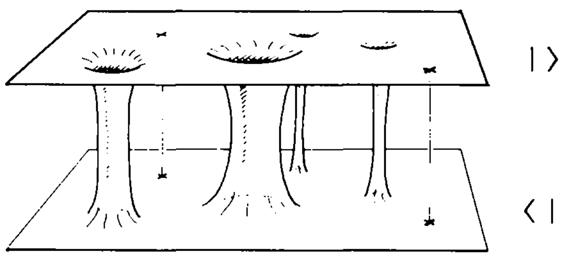

Fig. 7. Swiss cheese model of space-time with many particles. The big holes are black holes connecting bra and ket space. The thin lines are lighter particles, which in a pure state connect more or less the same points of bra and ket space.

matrix formalism because the pair of universes do nothing but represent the bra and ket states of Hilbert space. Of course if the Kruskal frames of each hole are saturated with matter these bra and ket universes remain disconnected and the metric (which could be referred to as "no-bra-metric") still represents a pure state.

The author owes much to a fruitful collaboration with T. Dray resulting in the essential work of refs. [8, 9].

References

[1] S.W. Hawking, Breakdown of predictability in gravitational collapse, Phys. Rev. D14 (1976) 2460; The unpredictability of quantum gravity, Cambridge Univ. preprint (May 1982)

[2] G. 't Hooft, Ambiguity of the equivalence principle and Hawking's temperature, J. Geom. Phys. 1 (1984) 45

[3] S.W. Hawking, Particle creation by black holes, Comm. Math. Phys. 43 (1975) 199;
J.B. Hartle and S.W. Hawking, Path-integral derivation of black hole radiance, Phys. Rev. D13 (1976) 2188;
W.G. Unruh, Notes on black-hole evaporation, Phys. Rev. D14 (1976) 870

[4] J.D. Bekenstein, Black holes and the second law, Nuovo Cim. Lett. 4 (1972) 737; Black holes and entropy, Phys. Rev. D7 (1973) 2333; Generalized second law of thermodynamics in black-hole physics, D9 (1974) 3292

[5] J.D. Bekenstein, Non existence of baryon number for static black holes, Phys. Rev. D5 (1972) 1239, 2403;
Ya.B. Zeldovich, A new type of radioactive decay: gravitational annihilation of baryons, Sov. Phys. JETP 45 (1977) 9;
A.D. Dolgor and Ya.B. Zeldovich, Cosmology and elementary particles, Rev. Mod. Phys. 53 (1981) 1

[6] S. Weinberg, in General relativity, an Einstein centenary survey, ed. S.W. Hawking and W. Israel (Cambridge Univ. Press, 1979) 795

[7] W.G. Unruh and R.M. Wald, What happens when an accelerating observer detects a Rindler particle, Berkeley preprint

[8] T. Dray and G. 't Hooft, The gravitational shock wave of a massless particle, Nucl. Phys. B.253 (1985) 173

[9] T. Dray and G. 't Hooft, Matter at the horizon of the Schwarzschild black hole, or A comment on the Schwarzschild black hole, Utrecht preprint (September 1984); to be published

[10] S.W. Hawking, private communication;
R.M. Wald, private communication;
D.J. Gross, Is quantum gravity unpredictable?, Nucl. Phys. B236 (1984) 349