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The Emergence of Quantum Mechanics

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Abstract

It is pointed out that a mathematical relation exists between cellular automata and quantum field theories. Although the proofs are far from perfect, they do suggest a new look at the origin of quantum mechanics: quantum mechanics may be nothing but a natural way to handle the statistical correlations of a cellular automaton. An essential role for the gravitational force in these considerations is suspected.

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1. Introduction.

Quantum Mechanics is usually treated as a theory for the dynamics of tiny systems such as atoms, subatomic particles, or alternatively weak fields over small regions of space and time, where it *replaces* classical Newtonian dynamics. Also the way logical arguments are supposed to be handled when dealing about "reality", is demanded to be subject to subtle but important changes.

After the dust settles, quantum mechanics is found to produce only statements about the average behavior of tiny systems, rather than any given individual system, as if individual systems have no right to have any notion of reality attached to them. Nevertheless, it is also stressed that the theory can be extremely accurate; it is much more than a set of fuzzy assertions for objects that are too small to be directly observed.

In numerous occasions, physicists have expressed their concern about this state of the theory, and attempts were made at improving this situation. Albert Einstein, Erwin Schrödinger, and later David Bohm and John Bell, among many others, tried their forces to replace quantum mechanics by something better.

In the present work, we do not attempt to replace quantum mechanics by something else. To the contrary, the present theory without any further amendments such as a collapse of the wave function or a many-world ontology, is accepted as a correct description of observations in the real world. However, we do stress that quantum mechanics does not describe reality directly. It meticulously quantifies what we see and measure in terms of equations, while any attempt to give a physical description of what we see, in terms of atoms, fields, subatomic particles, evolving universes and what not, is only decoration and does not refer to reality. We suspect that the theory does refer to reality, but that reality will have to be described in quite different physical terms.

Whatever the real objects are, quantum mechanics emerges as an extremely effective mathematical machinery to describe their statistical features in terms of probability distributions.

We start from "primordial" or "primitive" quantization, a procedure proposed by the author some time ago [1]—[6]. Imagine some set of classical dynamical equations of motion. One could think of the motion of the hands of a clock, but we can also imagine other completely classical objects varying from particles, strings or branes following classical equations of motion, to planets obeying Newton's laws. One then starts from a description of the entire space of all classical states, and subsequently promotes each and every one of these states to an element of the basis of a Hilbert space. The time evolution of these states from time t_1 to t_2 is then generated by an operator that we will call the evolution operator $U(t_1, t_2)$. If the original world that we wanted to describe is strictly continuous, the number of states in Hilbert space is non denumerable, and this sometimes causes complications in pursuing this program. Often a fundamentally discrete model can be used, which then makes our work easier. The evolution operator can then be constructed systematically.

If the system considered is *time reversible*, which means that the past can be re-

constructed from the future as easily as the future can be derived from the past, the evolution operator is also *unitary*, and this enables us to construct an other operator H, called 'Hamiltonian', such that

$$U(t_1,\, t_2) = e^{-i(t_2 - t_1)H} \ , \tag{1.1}$$

where H is hermitean, and its positivity and its symmetry structures can be studied. If the Hamiltonian that emerges from such calculations resembles the one used in familiar quantum theories, we have a mathematical system that on the one hand is based on an entirely classical model while on the other hand it allows for a description entirely in line with Quantum Mechanics.

Therefore, quantum theories of this sort could be interpreted as "hidden variable theories". The question then arises how to deal with Bell's inequalities[7][8]. These are inequalities that describe boundaries for measurements of physical features such as spin of quantum entangled objects. If the system is a classical one, the boundaries cannot be surpassed, whereas they are surpassed in a quantum theory. For many investigators, this is sufficient reason to categorically reject all hidden variable theories. However, the procedure just described appears to give us a number of interesting models that could perfectly well serve as good examples for 'hidden variables'. What is wrong with them? This is one of the questions we will have to answer. The situation is quite delicate and interesting. Our bottom line will be that it may well be possible to construct classical theories underlying Quantum Mechanics along these lines, but, quite remarkably, the gravitational force and General Relativity might be essential for a deeper understanding of the underlying structures.

For future reference, we will consider those degrees of freedam that describe the original classical states of the system as *beables*[9]. Beables $B(t)$ are operators that, at all times t, commute with all other beables:

$$[B(t_1),\, B(t_2)] = 0 \ , \qquad \forall\ t_1, t_2 \ . \tag{1.2}$$

A c*hangeable* $C(t)$ is an operator that maps a beable onto an other single beable:

$$C(t)\, B_1(t) = B_2(t)\, C(t) \ . \tag{1.3}$$

Finally, a *superimposable* S is an operator that can be any quantum superposition of any set of beables and/or changeables.

The eigenstates of the Hamiltonian defined in Eq. (1.1) will always be quantum superpositions. If we limit ourselves to low energy states only, this means that, from the start, we only talk of quantum entangled states. This is what distinguishes our models from other 'hidden variable' theories: we never attempt to reform a set of states such as the ones used in the 'Standard model', into totally classical states. Only the suspected underlying dynamical laws are classical, but the states considered are always quantum states, and usually highly entangled.

We shall not completely resolve the issue with the Bell inequalities, but since we do accept the emergence of the need for fully quantum mechanical states to describe probabilities, we suspect that these Bell inequalities will be a lot harder to use as a "no go" theorem for hidden variables than usually assumed.

2. The Hamiltonian of a deterministic system

According to the theory just described, we have discrete sets of physical data that evolve according to non-quantum mechanical, deterministic laws. Let us consider the case that also the time evolution is fundamentally discrete. The theory then can be defined in terms of an evolution operator U_0 that describes one step in time. Hence, we would like to write

$$(U_0)^k = e^{-iHk} \ , \tag{2.1}$$

where U_0 acts on the states $|n\rangle$ that are treated as basis vectors of a Hilbert space, so that H plays the role of a *quantum* Hamiltonian. We get conventional quantum mechanics if we can assure that H is bounded from below,

$$\langle\psi|H|\psi\rangle \geq \langle\psi_0|H|\psi_0\rangle \ , \tag{2.2}$$

for one state $|\psi_0\rangle$ and all states $|\psi\rangle$, where $|\psi\rangle$ and $|\psi_0\rangle$ may be any superposition of states $|n\rangle$, in particular any eigen state of H. In principle, an operator obeying this demand is not difficult to construct. Using the Fourier transform

$$x = \pi - \sum_{k=1}^{\infty} \frac{2}{k} \sin kx \ , \qquad 0 < x < 2\pi \ , \tag{2.3}$$

we derive:

$$H = \pi + \sum_{k=1}^{\infty} \frac{i}{k} (U_0^k - U_0^{-k}) \ . \tag{2.4}$$

This Hamiltonian has only eigenvalues between 0 and 2π, and it reproduces Eq. (2.1). The importance of having a lower bound of the Hamiltonian eigenvalues is that the lowest state can be identified as the 'vacuum state', and the first excited states can be interpreted as states containing particles. Thermodynamics gives us mixed states with probabilities

$$\varrho = C\, e^{-E/kT} \ , \tag{2.5}$$

However, in *extensive* systems, such as Fock space for a quantum field theory, this Hamiltonian is not good enough, for two (related) reasons. One is that the very high k contributions in Eq. (2.4) refer to large times, and this implies that these contributions are non-local. If interactions spread with the speed of light, the Hamiltonian will generate direct interactions over spatial distances proportional to k. This necessitates a cut-off: if the time steps are assumed to be one Planck time, then also k time steps may perhaps be considered to be short enough to reproduce local physics at the Standard Model scale. Using a cutoff in Eq. (2.4) gives an energy spectrum as sketched in Fig. 1. It is derived from the fact that, if a system has periodicity N, the eigenvalues of U_0 are $e^{2\pi i n/N}$.

In this figure, a smooth cut-off has been applied (cutting off the large k values with a Gaussian exponential). We see that, as a consequence, the lowest energy states are severely affected; their energy eigenvalues are now quadratic in the momenta k, so that,

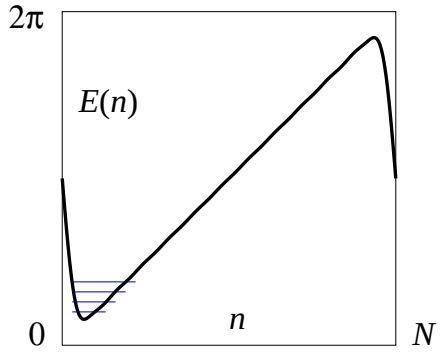

Figure 1: The energy spectrum $E(n)$ after a cutoff at large k. The lowest energy states (small lines) are severely affected by the cut-off.

here, the Hamiltoniann does not reproduce the correct evolution operator (2.1). This region, however, is important when applying the laws of thermodynamics, since these states dominate in the Boltzmann expression (2.5).

The second problem is that we expect a Hamiltonian to decouple when states are considered that are spatially separated: $H = H_1 + H_2$. In that case, one cannot maintain that the eigenvalues of the entire Hamiltonian stay within the bounds $(0,\, 2\pi)$. We return to this point in Section 5.

3. A Cellular Automaton

The construction of an extensive Hamiltonian was suggested in [6]. Consider a cellular automaton. Space and time[10] are both discrete here: we have a D dimensional space, where positions are indicated by integers: $\vec{x} = (x^1,\, x^2,\, \cdots,\, x^D)$, where $x^i \in \mathbb{Z}$. Also time t will be indicated by integers, and time evolution takes place stepwise. The physical variables $F(\vec{x},\, t)$ in the model could be assumed to take a variety of forms, but the most convenient choice is to take these to be integers modulo some integer $\mathbb{N}$. We now write down an explicit model, where these physical degrees of freedom are defined to be attached only to the even lattice sites:

$$\sum_{i=1}^{D} x^i + t \;=\; \text{even}. \tag{3.1}$$

Furthermore, the data can be chosen freely at two consecutive times, so for instance at $t = 0$, we can choose the initial data to be $\{F(\vec{x},\; t = 0),\; F(\vec{x},\; t = 1)\}$.

The dynamical equations of the model can be chosen in several ways, provided that they are time reversible. To be explicit, we choose them to be as follows:

$$
\begin{aligned}
&F(\vec{x},\, t+1) \;=\; F(\vec{x},\, t-1) + \\
&Q\left(F(x^1 \pm 1, x^2, \cdots, x^D,\, t),\, \cdots,\, F(x^1, \cdots, x^D \pm 1,\, t)\right) \;\text{Mod}\; \mathbb{N}\;, \\
&\text{when} \quad \textstyle\sum_i x^i + t \quad \text{is odd}\,,
\end{aligned}
\tag{3.2}
$$

where the integer Q is some arbitrary given function of all variables indicated: all nearest neighbors of the site $\vec{x}$ at time t. This is time reversible because we can find $F(\vec{x},\, t-1)$ back from $F(\vec{x},\, t+1)$ and the neighbors at time t. Assuming Q to be a sufficiently irregular function, one generally obtains quite non-trivial cellular automata this way. Indeed, this category of models have been shown to contain examples that are computationally universal [11]. Models of this sort are often considered in computer animations.

We now discuss the mathematics of this model using Hilbert space notation. We switch from the Heisenberg picture, where states are fixed, but operators such as the beables $F(\vec{x},\, t)$ are time dependent, to the Schrödinger picture. Here, we call the operators F on the even sites $X(\vec{x})$, and the ones on the odd sites $Y(\vec{x})$. As a function of time t, we alternatingly update $X(\vec{x})$ and $Y(\vec{x})$, so that we construct the evolution operator over two time steps. Keeping the time parameter t even:

$$
U(t,\, t-2) = A \cdot B\;, \tag{3.3}
$$

where A updates the data $X(\vec{x})$ and B updates the data $Y(\vec{x})$.

Updating the even sites only, is an operation that consists of many parts, each defined on an even space coordinate $\vec{x}$, and all commuting with one another:

$$
A = \prod_{\vec{x}\ \text{even}} A(\vec{x})\;, \quad [A(\vec{x}),\, A(\vec{x}\,')] = 0\;, \tag{3.4}
$$

whereas the B operator refers only to the odd sites,

$$
B = \prod_{\vec{x}\ \text{odd}} B(\vec{x})\;, \quad [B(\vec{x}),\, B(\vec{x}\,')] = 0\;. \tag{3.5}
$$

Note however, that the operators $A(\vec{x})$ and $B(\vec{x}\,')$ do not all commute. If $\vec{x}$ and $\vec{x}\,'$ are neighbors, then

$$
\vec{x} - \vec{x}\,' = \vec{e}\,, \quad |\vec{e}| = 1 \quad \rightarrow \quad [A(\vec{x}),\, B(\vec{x}\,')] \neq 0\;. \tag{3.6}
$$

It is important to observe here that both the operators $A(\vec{x})$ and $B(\vec{x})$ only act in finite subspaces of Hilbert space, and they are all unitary, so we can easily write them as follows:

$$
A(\vec{x}) = e^{-i\mathfrak{a}(\vec{x})}\;, \qquad B(\vec{x}) = e^{-i\mathfrak{b}(\vec{x})}\;. \tag{3.7}
$$

Note that $A(\vec{x})$ and $B(\vec{x})$ are changeables, while $\mathfrak{a}(\vec{x})$ and $\mathfrak{b}(\vec{x})$ will be superimposables, in general, and they are hermitean. We can write

$$
\mathfrak{a}(\vec{x}) = \mathcal{P}_x(\vec{x})\; Q(\{Y\})\;, \quad \mathfrak{b}(\vec{x}) = \mathcal{P}_y(\vec{x})\; Q(\{X\})\;, \tag{3.8}
$$

where $\mathcal{P}_x(\vec{x})$ is the generator for a one-step displacement of $X(\vec{x})$:

$$e^{i\mathcal{P}_x(\vec{x})}|X(\vec{x})\rangle \stackrel{\text{def}}{=} |X(\vec{x}) - 1 \text{ Mod } \mathbb{N}\rangle \ , \tag{3.9}$$

and, similarly, $\mathcal{P}_y(\vec{x})$ generates one step displacement of the function $Y(\vec{x})$.

As an example, we give the matrix P for the case $\mathbb{N} = 5$. Defining the numerical coefficients $\alpha = 2\sin(\pi/5) + \sin(2\pi/5)$ and $\beta = 2\sin(2\pi/5) - \sin(\pi/5)$, we have

$$P = \frac{4\pi i}{25}\begin{pmatrix} 0 & -\alpha & \beta & -\beta & \alpha \\ \alpha & 0 & -\alpha & \beta & -\beta \\ -\beta & \alpha & 0 & -\alpha & \beta \\ \beta & -\beta & \alpha & 0 & -\alpha \\ -\alpha & \beta & -\beta & \alpha & 0 \end{pmatrix} \ ; \qquad e^{iP} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{3.10}$$

We see that

$$[\mathfrak{a}(\vec{x}),\, \mathfrak{a}(\vec{x}\,')] = 0 \ , \quad [\mathfrak{b}(\vec{x}),\, \mathfrak{b}(\vec{x}\,')] = 0 \ , \quad \forall\,(\vec{x},\, \vec{x}\,') \ ; \tag{3.11}$$

$$[\mathfrak{a}(\vec{x}),\, \mathfrak{b}(\vec{x}\,')] = 0 \quad \text{only if} \quad |\vec{x} - \vec{x}\,'| > 1 \ . \tag{3.12}$$

A consequence of Eqs. (3.11) is that also the products A in Eq. (3.4) and B in Eq. (3.5) can be written as

$$A = e^{-i\sum_{\vec{x}\ \text{even}}\mathfrak{a}(\vec{x})} \ , \qquad B = e^{-i\sum_{\vec{x}\ \text{odd}}\mathfrak{b}(\vec{x})} \ . \tag{3.13}$$

However, now A and B do not commute. Nevertheless, we wish to compute the total evolution operator U for two consecutive time steps, writing it as

$$U = A \cdot B = e^{-i\mathfrak{a}}\, e^{-i\mathfrak{b}} = e^{-2iH} \ . \tag{3.14}$$

For this calculation, we could use the power expansion given by the Baker-Campbell-Hausdorff formula[12],

$$\begin{aligned} &e^P\, e^Q = e^R \ , \\ &R = P + Q + \tfrac{1}{2}[P,Q] + \tfrac{1}{12}[P,[P,Q]] + \tfrac{1}{12}[[P,Q],Q] + \tfrac{1}{24}[[P,[P,Q]],Q] + \cdots , \end{aligned} \tag{3.15}$$

a series that continues exclusively with commutators[12]. Replacing P by $-i\mathfrak{a}$, Q by $-i\mathfrak{b}$ and R by $-2iH$, we find a series for the 'hamiltonian' H in the form of an infinite sequence of commutators. Now note that the commutators between the local operators $\mathfrak{a}(\vec{x})$ and $\mathfrak{b}(\vec{x}\,')$ are non-vanishing only if $\vec{x}$ and $\vec{x}\,'$ are neighbors, $|\vec{x} - \vec{x}\,'| = 1$. Consequently, if we insert the sums (3.13) into Eq. (3.15), we obtain again a sum:

$$\begin{aligned} &H = \sum_{\vec{x}} \mathcal{H}(\vec{x}) \ , \\ &\mathcal{H}(\vec{x}) = \tfrac{1}{2}\mathfrak{a}(\vec{x}) + \tfrac{1}{2}\mathfrak{b}(\vec{x}) + \mathcal{H}_2(\vec{x}) + \mathcal{H}_3(\vec{x}) + \cdots \ , \end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\mathcal{H}_2(\vec{x}) &= -\tfrac{1}{4}i\sum_{\vec{y}}[\mathfrak{a}(\vec{x}),\,\mathfrak{b}(\vec{y})]\ ,\\ \mathcal{H}_3(\vec{x}) &= -\tfrac{1}{24}\sum_{\vec{y}_1,\,\vec{y}_2}[\mathfrak{a}(\vec{x})-\mathfrak{b}(\vec{x})\,,\ [\mathfrak{a}(\vec{y}_1),\mathfrak{b}(\vec{y}_2)]]\ ,\quad \text{etc.}\end{aligned} \tag{3.17}$$

All these commutators are only non-vanishing if the coordinates $\vec{y}$, $\vec{y}_1$, $\vec{y}_2$, etc., are all neighbors of the coordinate $\vec{x}$. It is true that, in the higher order terms, next-to-nearest neighbors may enter, but still, one may observe that these operators are all local functions of 'field operators' $\Phi(\vec{x},\,t)$, and thus we arrive at a hamiltonian H that can be regarded as the sum over D-dimensional space of a Hamilton density $\mathcal{H}(\vec{x})$, which has the property that

$$[\mathcal{H}(\vec{x}),\,\mathcal{H}(\vec{x}\,')]=0\ ,\quad \text{if}\quad |\vec{x}-\vec{x}\,'|\gg 1\ . \tag{3.18}$$

The $\gg$ symbol here means that at the n^{th} order in the BCH series, $\vec{x}$ and $\vec{x}\,'$ must be further than n steps away from one another.

At every finite order of the series, the Hamilton density $\mathcal{H}(\vec{x})$ is a finite-dimensional Hermitean matrix, and therefore, it will have a lowest eigenvalue h. In a large but finite volume V, the total hamiltonian H will therefore also have a lowest eigenvalue, obeying

$$E_0 > h\,V\ . \tag{3.19}$$

The associated eigenstate $|\,0\rangle$ might be identified with the 'vacuum'. This vacuum is stationary, even if the automaton itself may have no stationary solution. The next-to-lowest eigenstate may be a one-particle state. In a Heisenberg picture, the fields $F(\vec{x},t)$ may create a one-particle state out of the vacuum. Thus, we arrive at something that resembles a genuine quantum field theory. The states are quantum states in complete accordance with a Copenhagen interpretation. The fields $\mathfrak{a}(\vec{x},t)$ and $\mathfrak{b}(\vec{x},t)$ should obey the Wightman axioms.

There are three ways, however, in which this theory differs from conventional quantum field theories. One is, of course, that space and time are discrete. Well, maybe there is an interesting 'continuum limit', in which the particle mass(es) is(are) considerably smaller than the inverse of the time quantum.

Secondly, no attempt has been made to arrive at Lorentz invariance, or even Gallilei invariance. Thus, the dispersion relations for these particles, if they obey any at all, may be nothing resembling conventional physical particles. Do note, however, that no physical information can travel faster than velocity one in lattice units. This is an important constraint that the model still has in common with special relativity.

But the third difference is more profound. It was tacitly assumed that the Baker-Campbell-Hausdorff formula converges. This is often not the case. In Section 4, we shall argue that the series will converge well only if sandwiched between two eigenstates $|E_1\rangle$ and $|E_2\rangle$ of H, where E_1 and E_2 are the eigenvalues, that obey

$$2|E_1-E_2|<2\pi\hbar/\Delta t\ , \tag{3.20}$$

where Δt is the time unit of our clock, and the first factor 2 is the one in Eq. (3.14). ("Planck's constant", $\hbar$, has been inserted merely to give time and energy the usual physical dimensions.)

This may seem to be a severe restriction, but, first, one can argue that $2\pi\hbar/\Delta t$ here is the Planck energy, and in practice, when we do quantum mechanics, we only look at energies, or rather energy differences, that indeed are much smaller than the Planck energy. Does this mean that transitions with larger energy differences do not occur? We must realize that energy is perhaps not exactly conserved in this model. Since time is discrete, energy at first sight seems to be only conserved modulo π, and this could indicate that our 'vacuum state' is not stable after all. The energy might jump towards other states by integer multiples of π. In Section 4 however, we argue that such violations of energy conservation will *not* occur, and the existence of an hamiltonian density is a more profound property of all cellular automata that allow time reversal (so that the evolution is obviously unitary).

The conclusion we are able to draw now, is that procedures borrowed from genuine quantum mechanics can be considered, and they may lead to a rearrangement of the states in such a way that beables, changeables and superimposables naturally mix, leaving an effective description of a system at large time and distance scales for which only quantum mechanical language applies. This, we think, is all we really need to understand why it is quantum mechanics that seems to dominate the world of atoms and other tiny particles, which, though small compared to humans, are still very large compared to the Planck scale.

At this point we like to remark that there is a precedent. Quantum field theory in fact can be used conveniently to solve a completely classical problem: the 2 dimensional Ising Model[13].

4. Convergence of the BCH expansion

When the operators A and B are bounded below and above, the BCH series expansion is expected to have a finite radius of convergence, but it certainly does not converge in general. To understand the situation, let us consider a quick derivation of the expansion. Given the operators A and B, consider the definition of an operator $C(\sigma)$ as a continuous function of σ, obeying

$$e^{iC(\sigma)} = e^{i\sigma A} e^{iB} \ ; \qquad C(0) = B \ . \tag{4.1}$$

Differentiating with respect to σ gives

$$\int_0^1 \mathrm{d}x \, e^{ixC(\sigma)} \, \frac{\mathrm{d}}{\mathrm{d}\sigma} C(\sigma) \, e^{-ixC(\sigma)} = A \ ; \tag{4.2}$$

Diagonalizing $C(\sigma)$ at a given point $\sigma = \sigma_0$, we define

$$C(\sigma_0)|E\rangle \stackrel{\text{def}}{=} E|E\rangle \ , \qquad \langle E_1| \, \frac{\mathrm{d}}{\mathrm{d}\sigma} F(\sigma) \, |E_2\rangle \Big|_{t=t_0} \stackrel{\text{def}}{=} F'_{12} \ . \tag{4.3}$$

From Eq. (4.2) one then derives

$$\begin{aligned} C'_{12} &= \frac{i(E_1-E_2)}{e^{i(E_1-E_2)}-1}\,A_{12} \;=\; \sum_{n=0}^{\infty}\frac{i^n B_n}{n!}(E_1-E_2)^n\,A_{12} \\ &= \sum_{n=0}^{\infty}\frac{i^n B_n}{n!}[C,[C,\cdots,[C,A]]\cdots]_{12}\;, \end{aligned} \tag{4.4}$$

where B_n are the Bernouilli numbers. Recursively, this defines $C(\sigma)$. It is clear that this series (4.4) converges precisely when

$$|E_1-E_2|<2\pi\;. \tag{4.5}$$

Note that this does not imply that the BCH series itself converges when sandwiched between two states $|E_1\rangle$ and $|E_2\rangle$ that are less than 2π apart, because the condition (4.5) must hold at all σ, while the states $|E_i\rangle$ are σ-dependent. The derivation merely suggests that, if during the entire calculation, only states are considered whose energies are much less separated than 2π in natural units, *at all stages*, the series may be expected to converge. This condition may nevertheless be considered to be a weak one, likely to be fulfilled in many cases, because the natural unit here is the Planck unit, $E_{\rm Planck}/c^2\approx 21\mu$g, which is always much larger than any experiment done in quantum mechanics.

An interesting aside is the observation that the Baker-Campbell-Hausdorff expansion (3.15) can be rewritten in terms of a series that contains much fewer terms. Write

$$-i\mathfrak{a}=P+Q\;,\qquad -i\mathfrak{b}=P-Q\;. \tag{4.6}$$

and note that we are really only interested in the conjugacy classes of H, not H itself:

$$e^{P+Q}\,e^{P-Q}=e^{F}\,e^{R}\,e^{-F}\;, \tag{4.7}$$

where F can be chosen with certain amounts of freedom. Noting that interchanging $\mathfrak{a}$ and $\mathfrak{b}$ should give us a Hamiltonian that is just as good as H, and certainly in the same conjugacy class, we search for an F such that

$$R(P,Q)=R(P,-Q)\;,\qquad R(-P,Q)=-R(P,Q)\;. \tag{4.8}$$

Using the short hand notation $QP^3Q = [Q,[P,[P,[P,Q]]]]$, etc., one finds

$$\begin{aligned} R \;=\;& 2P-\tfrac{1}{12}QPQ+\tfrac{1}{960}Q(8P^2-Q^2)PQ+ \\ & \tfrac{1}{60480}Q(-51\,P^4-76\,QPQP+33\,Q^2P^2+44\,PQ^2P-\tfrac{3}{8}\,Q^4)PQ \\ & +\mathcal{O}(P,Q)^9\;. \end{aligned} \tag{4.9}$$

Already the third term in this expression goes beyond the expansion (3.15), but (4.9) does not converge much faster.

5. Quantum Gravity

To get quantum particles that have a dispersion law:

$$E(p) \to \frac{p^2}{2m} \ , \tag{5.1}$$

one needs at least some form of Galilean invariance — so that particles can have a velocity. Since cellular automata have a limiting speed of the transfer of information, this would have to be replaced by Lorentz invariance from the start, so, special relativity is hard to avoid even at the early stages of constructing models. However, both the Galilei group and the Lorentz group are *non compact*, and this makes it very difficult to turn these symmetries into even approximately reasonable symmetries for cellular automata[1].

Not only special relativity, but also *General Relativity* may perhaps not be left out before obtaining a true understanding of Quantum Mechanics as a description of the statistical features of a classical system. We saw in our calculations that, first, the construction (3.16) for our Hamiltonian would fail to produce an extensive expression for the energy. The total energy of the Universe would always be less than $2\pi/T_{\rm Planck}$ (For simplicity, we take our fundamental time unit to be the Planck time). Yet, we wish to describe widely separated regions $(1,\ 2,\ 3, \cdots)$ of the Universe in terms of a total Hamiltonian $H_{\rm tot}$ that is approximately written as

$$H_{\rm tot} = H_1 + H_2 + H_3 + \cdots \ , \tag{5.2}$$

so that, inevitably, the total energy can become much larger than the limit $2\pi/T_{\rm Planck}$.

Our treatment of the cellular automaton conveniently produced for us a Hamilton density, so that the global energy is extensive, but divergence of the CBH series forced us to limit ourselves to quantum states whose energies do stay closer together than the Planck energy.

It would be much better if a procedure could be found where such limitations do not have to be imposed. This would be the case if we had another way to define Hamilton density. This is where General Relativity comes in. When the gravitational force is taken into account, we may have a space-dependent gravitational potential, which leads to space-dependent dynamical operators $\tau(\vec{x})$ that determine the local clock speed. Thus, the evolution operator becomes

$$U(\tau(\vec{x})) \ = \ e^{-i\int \tau(\vec{x})\mathcal{H}(\vec{x}){\rm d}^3\vec{x}} \ . \tag{5.3}$$

Thus, such a theory allows for a direct definition of Hamilton densities. By comparing the evolution operators at different gravitational potentials, one derives $\mathcal{H}(\vec{x})$. In General Relativity,

$$\tau(\vec{x}) = t\sqrt{-g^{00}(\vec{x})} \ . \tag{5.4}$$

[1] One could have hoped that an approximate symmetry turns into an exact symmetry in the continuum limit.

6. Conclusions

The importance of these calculations is that each of the commutators in the BCH series are non vanishing only if they consist of neighboring operators; therefore, the resulting Hamiltonian can be written as the sum of Hamilton densities. Therefore, it seems as if the Hamiltonian is constructed just as in a quantized field theory. Distant parts of this 'universe' evolve independently. The total Hamiltonian has eigen values much greater than 2π, and are all bounded from below. If now we concentrate on the lowest lying states, we see that these consist of localized particles resembling what we see in the real world. They have positive energies, so that thermodynamics applies to them.

But, as we saw, there is a caveat: the BCH series (3.15) does not converge. To be precise, the operator H is ambiguous when two of its eigenvalues get further separated than $2\pi/T_{\rm Planck}$, as can be seen directly from its definition (2.1). That is where the series' radius of convergence ends.

One must deduce that the series (3.15) cannot be truncated easily; it will have to be resummed carefully, and whether or not a resummation exists remains questionable. We can argue that these difficulties only refer to matrix elements between states whose energy eigenvalues are more than the Planck energy apart, which is a domain of quantum physics that has never been addressed experimentally anyway, so maybe they can safely be ignored. Yet this situation is unsatisfactory, so more work is needed.

We do conclude with an important conjecture: perhaps our world is not quantum mechanical, but only our perception of it.

References

[1] G. 't Hooft, "Quantum Gravity as a Dissipative Deterministic System", *Class. Quant. Grav.* **16** (1999) 3263, arXiv:gr-qc/9903084.

[2] G. 't Hooft, "Determinism in Free Bosons", *Int. J. Theor. Phys.* **42** (2003) 355, arXiv:hep-th/0104080.

[3] H. Th. Elze, "Deterministic models of quantum fields", *J. Phys.*: Conf. Ser. **33** (2006) 399, arXiv:gr-qc/0512016v1.

[4] M. Blasone, P. Jizba and H. Kleinert, *Annals of Physics* **320** (2005) 468, arXiv: quant-ph/0504200; id., *Braz. J. Phys.* **35** (2005) 497, arXiv: quant-ph/0504047.

[5] G. 't Hooft, "Emergent quantum mechanics and emergent symmetries", presented at PASCOS 13, Imperial College, London, July 6, 2007; ITP-UU-07/39, SPIN-07/27; arXiv:hep-th/0707.4568.

[6] G. 't Hooft, "Entangled quantum states in a local deterministic theory", 2nd Vienna Symposium on the Foundations of Modern Physics (June 2009), ITP-UU-09/77, SPIN-09/30; arXiv:0908.3408v1 [quant-ph].

[7] A. Einstein, B. Podolsky and N. Rosen, "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?", *Phys. Rev.* **47** (1935) 777.

[8] J. S. Bell, "Speakable and unspeakable in quantum mechanics" (Cambridge Univ. Press, Cambridge, 1987).

[9] G. 't Hooft, "The mathematical basis for deterministic quantum mechanics", in *Beyond the Quantum*, World Scientific, Th. M. Nieuwenhuizen et al, ed., pp.3-19, `arXiv: quant-ph/0604008`;

[10] A.P. Balachandran and L. Chandar, *Nucl. Phys.***B 428** (1994) 435;

[11] D.B. Miller and E. Fredkin, "Two-state, Reversible, Universal cellular Automata in Three Dimensions", Proc. 2nd Conf. on Computing Frontiers, Ischia, Italy: ACM 45, doi: 10.1145/1062271, `arXiv:nlin/0501022`.

[12] A.A. Sagle and R.E. Walde, "Introduction to Lie groups and Lie Algebras", Academic Press, New York, 1973. ISBN 0-12-614550-4.

[13] B. Kaufman, *Phys. Rev.* **76** (1949) 1232; B. Kaufman and L. Onsager, *Phys. Rev.* **76** (1949) 1244.