1. Green function

Equations:

$$\begin{aligned}\Psi_{rr}+\frac{1}{r}\Psi_r-\Psi_{tt} &= 0\ ;\\ \gamma_r &= r(\Psi_r^2+\Psi_t^2)\ ,\\ \gamma_t &= 2r\Psi_r\Psi_t\ .\end{aligned} \tag{1.1}$$

Given arbitrary values for $\Psi(r,0)$ and $\Psi_t(r,0)$, the solution obeys

$$\Psi(x,t)=\int_0^\infty \mathrm{d}r(G(x,r,t)\Psi_t(r,0)+G_t(x,r,t)\Psi(r,0))\ ,\qquad t>0 \tag{1.2}$$

where G is

$$G(x,r,t)=\frac{r}{2\pi}\oint_{\sqrt{+}}\mathrm{d}\varphi\frac{1}{\sqrt{2xr\cos\varphi+t^2-x^2-r^2}}\ , \tag{1.3}$$

where the symbol $\sqrt{+}$ indicates that the integral should only be taken over the values of φ for which the entry of the square root is positive. If $t<|x-r|$ there are no real φ values obeying this, so G vanishes there.

Derivation: take action functional S in 3+1 dimensions:

$$S=\int \mathrm{d}^3\vec{x}\mathrm{d}t\left(-\tfrac{1}{2}(\vec{\partial}\Psi)^2+\tfrac{1}{2}\Psi_t^2+J(\vec{x},t)\Psi\right)\ , \tag{1.4}$$

leading to the field equation

$$\Psi_{tt}-\vec{\partial}^2\,\Psi=J(\vec{x},t)\ . \tag{1.5}$$

General solution:

$$\Psi(\vec{x},t) = \int \mathrm{d}^3\vec{x}'\int_0^\infty \mathrm{d}\tau\, G(\vec{x}-\vec{x}',\,\tau)J(\vec{x}',\,t-\tau)\ , \tag{1.6}$$

$$G(\vec{x},\,\tau) = \frac{1}{4\pi r}\,\delta(r-\tau)\,\theta(\tau)\ ,\qquad r^2=\vec{x}^{\,2}\ . \tag{1.7}$$

G is the so-called Green function of the system. Now take the cylindrically symmetric case,

$$J=J(r,t)\ ,\qquad \Psi=\Psi(r,t)\ , \tag{1.8}$$

where

$$\frac{S}{2\pi}=\int_0^\infty \mathrm{d}r\int \mathrm{d}t\left(-\tfrac{1}{2}r\Psi_r^2+\tfrac{1}{2}r\,\psi_t^2+r\,J\,\Psi\right)\ , \tag{1.9}$$

in which case the field equation becomes ours with source term inserted:

$$\Psi_{tt}-\Psi_{rr}-\frac{1}{r}\Psi_r=J(r,t)\ . \tag{1.10}$$

Using (1.6) and (1.7), we find

$$\Psi(r_1,t) \;=\; \int_0^\infty \mathrm{d}r \oint \mathrm{d}\varphi \int \mathrm{d}z \int_0^\infty \mathrm{d}\tau \frac{r}{4\pi\tau}\,\delta(\varrho-\tau)\,J(r,\,t-\tau)\;, \qquad (1.11)$$

$$\text{where} \qquad \varrho \;\equiv\; \sqrt{z^2+r^2+r_1^2-2r\,r_1\cos\varphi}\;. \qquad (1.12)$$

The delta function can be rewritten as

$$\delta(\varrho-\tau) = \frac{\tau}{z}\,\delta\left(z-\sqrt{\tau^2+2r\,r_1\cos\varphi-r^2-r_1^2}\right)\;. \qquad (1.13)$$

If the square root exists there are two solutions for z, so

$$\Psi(r_1,\,t) \;=\; \int_0^\infty \mathrm{d}r \oint_{\sqrt{\mp}} \mathrm{d}\varphi \int_0^\infty \mathrm{d}t\, G(r_1,r,\tau)\,J(r,t-\tau) \qquad (1.14)$$

where G is as given in (1.3). The equation (1.2) is obtained by choosing $\psi(r,t)=0$ if $t<0$ and a source

$$J(r,t) = \delta(t)\Phi_2(r)+\delta'(t)\Phi_1(r)\;, \qquad (1.15)$$

so that the field equation (1.10) forces

$$\Psi(r,t)\to\Phi_1(r)\;,\qquad \Psi_t(r,t)\to\Phi_2(r)\;, \qquad (1.16)$$

$$\text{as} \qquad t\downarrow 0\;. \qquad (1.17)$$

To see that (1.16) indeed follows from (1.15) and (1.10), evaluate G for $t\ll r,r_1$:

$$G(x,r,t)\to\tfrac{1}{2}(\theta(r-x+t)-\theta(r-x-\tau))\;. \qquad (1.18)$$

To see that (1.3) indeed satisfies the field equation

$$G_{tt}-G_{xx}-(1/x)G_x = \delta(x-r)\delta(t) \qquad (1.19)$$

is an elementary exercise. It follows from the construction, and the direct argument is as follows: the integrand,

$$g(x,t,\varphi) = \frac{1}{\sqrt{2xr\cos\varphi+t^2-x^2-r^2}}$$

obeys the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}-\frac{\partial^2}{\partial t^2}+\frac{1}{x^2}\frac{\partial^2}{\partial\varphi^2}\right)g(x,t,\varphi)=0\;, \qquad (1.20)$$

so that integration of Eq. (1.19) over φ only gives boundary terms. To see the contribution of these boundary terms, reexpress the integral in terms of a closed contour integral in the complex plane. One finds that G can also be written as

$$G(x,r,t) = \frac{r}{2\pi}\int_{\sqrt{\mp}} \mathrm{d}\varphi\,\frac{1}{\sqrt{2x\,r\cosh\varphi+x^2+r^2-t^2}}\;, \qquad (1.21)$$

where the integral runs from $-\infty$ to ∞ or over the domain where the square root exists. The field equation then leads to an integral over φ of a function that is a pure derivative of a function of φ with appropriate boundary conditions. Either in Eq. (1.3) or in Eq. (1.21), we see that the boundaries vanish, unless $r = x$ and $t = 0$. The delta functions in (1.19) are deduced by inspecting the points where G is singular. Expression (1.3) ensures that $G = 0$ as soon as $|x - r| > t$.

This completes the proof that these solutions are completely causal. It is an elementary exercise in mathematical physics.