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HILBERT SPACE IN DETERMINISTIC THEORIES*

A Reconsideration of the Interpretation of Quantum Mechanics

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1. Conway's Game of Life

The prototype of a *deterministic* model of the universe is a toy that emerged in the early days of the personal computer, called *Conway's Game of Life*[1]. An infinite two dimensional array of cells is considered, where each cell carries a variable σ that may have the value 1 ("alive") or 0 ("dead"). At the beat of a clock, measuring time t in integers, life is spread over the array of cells, in accordance with a strict rule of evolution. The parameter in each cell at time t is being updated, depending on the value it and its 8 nearest neighbors had at time $t-1$. The rules are as follows:

1. Any live cell with fewer than two live neighbors dies, "as if by loneliness".
2. Any live cell with more than three live neighbors dies, "as if by overcrowding".
3. Any live cell with two or three live neighbors lives unchanged, to the next generation.
4. Any dead cell with exactly three live neighbors comes to life.

An extensive literature emerged on this model. One finds that the evolution of some large agglomeration of living cells can be quite complex. Some configurations, when surrounded by empty space, can reproduce themselves while moving along, in horizontal, vertical, or diagonal directions. Usually, a system dies after some time, but that may take very long.

From a physical point of view, the system has two main characteristics: *a*) it is *deterministic*; there is never any uncertainty as to what happens after any finite number of steps, but *b*) there is *information loss*; this means that, although the future follows unambiguously from the past, the converse is not true. A given configuration can emerge from many different initial configurations. We can call this "dissipation". One might find it "ugly", but one could also argue that this feature produces some enrichment, producing order even if one starts with an apparently random configuration at $t=0$.

We argue in this lecture that, in spite of appearances to the contrary, the universe we live in may have much in common with Conway's game of life. There might be determinism as well as information loss at the Planck scale of 10^{-33} cm. What then remains to be explained is how it can be possible that such a world exhibits typically quantum mechanical behavior at much larger scales, typically that of the Standard Model, at some 10^{-16} cm. Where do interference effects come from? Why is quantum mechanics time reversal symmetric and unitary? Why should one expect information loss at all at the Planck scale?

What we call quantum mechanics is the theory stating that the dynamical laws of nature can be formulated in the following way. We have a *Hilbert space*, $\mathcal{H}$, and all possible configurations one can encounter in the physical world at a given time t can be viewed as normalized elements $|\psi(t)\rangle$ of this Hilbert space. These elements then evolve according to a linear equation called Schrödinger equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = -iH|\psi(t)\rangle \;, \tag{1.1}$$

where the Hamiltonian H can be *any* linear Hermitian operator acting in $\mathcal{H}$, and whenever any observation or measurement is performed, one compares the evolving states with a template of elements $|\psi_a\rangle$ of $\mathcal{H}$, where a is some label. As postulated by Max Born, the quantity

$$W_a(t) = |\langle\psi_a|\psi(t)\rangle|^2 \tag{1.2}$$

is then the *probability* that the outcome of the measurement is the state a.

When devising a theory of nature, all one has to do is two things: 1) give a proper description of all possible states a of Hilbert space, which will usually form an orthonormal basis of $\mathcal{H}$. Subsequently, 2) one has to identify the Hamiltonian H. Although in principle *all* laws of nature should be phrased in this fashion, the peculiarities of quantum mechanics are most manifest in the domain of atomic and sub-atomic phenomena.

2. Quantum mechanics as a tool

Quantum mechanics is not only a theory describing phenomena at the atomic and sub-atomic scale. It is also a mathematical tool, useful for doing statistical calculations. The most elegant example of this is the *Two-dimensional Ising Model.* The mathematical question that was addressed here has nothing to do with quantum mechanics. Consider a large but finite two dimensional grid, filled with ones and zeros. We count not only the total number of ones as opposed to zeros, defining a ratio between 0 and 1, but we also count the number of borders between direct neighbors that are identical (a 1 adjacent to a 1, or a 0 adjacent to a 0), as opposed to the number of borders where the neighbors are different (a 1 adjacent to a 0). The mathematical question to be asked is:
How many ways are there to fill the grid such that these two ratios are fixed by some given numbers?
The problem looks totally unsolvable at first sight, but the remarkable discovery made by L. Onsager[2] is that, if the first of these numbers is given as $^1/_2$, so that the number of ones and zeros are equal, while the other number can be anything, then an exact solution can be obtained in the limit where the grid is very large.

The method that can be used to arrive at this solution was beautifully spelled out by Kaufman[2] in 1949. Indeed, essential use was made of *quantum* field theory. The transfer matrix acts in a genuine Hilbert space of states, and can be regarded as a quantum evolution operator, being the exponent of a Hamiltonian. The Ising model turns into a world of fermionic particles that, in this particular case, are free of any interaction, and so the exact set of eigenvalues of the Hamiltonian could be found. Quantum mechanics came out of the blue, as a tool rather than a theory. Could quantum mechanics as we know it in particle physics, not also be a tool? Could it be that we are dealing with a problem of statistics at very tiny scales, where only quantum mechanical methods are suitable to obtain precise though statistical information valid at much larger scales? We find this far from implausible. The idea is to be further investigated.

Quantum mechanics appears to be the answer to a problem — but what exactly is the problem? The views as sketched here will appear elsewhere in a much more elaborate description.

3. Beables and changeables

Quantum mechanics is a marvellous statistical description of the world as we experience it, but it appears not to be suitable to describe 'reality'. We have learned how to use quantum mechanics to make our predictions as accurate as possible, but we have not yet understood what the real world is that it describes. From a purely philosophical point of view, this situation is understandable and acceptable: we see things without truly understanding them; sometimes things happen in a highly predictable fashion, and sometimes, for instance when a given uranium atom decays, we are surprised by something we completely fail to foresee. Evidently, physics as a science is not finished; we are further away from the truth than many of us want us to believe.

How could it happen that the observed statistical features end up being quantum mechanical ones? This is the great question that we will investigate, but we will argue that quantum mechanics is not at all such a weird result as it appears to be for many. To see this, we are forced to consider some simplified models. Some of our models will be far *too* simplified, but hang on.

Our first model, to be called the Cogwheel Model, is the following. At every beat of a clock, a cogwheel with N teeth, and a mark on one of its teeth, rotates over an angle $2\pi/N$. After N beats, the mark is back at its starting position. In fact, this is a model representing any system that has N states that evolve with discrete time steps and is periodic with a period of N units of time. Labelling the states as $|0\rangle, \cdots, |N-1\rangle$, the evolution law is:

$$\begin{aligned} t &\rightarrow t+1 \\ |n\rangle &\rightarrow |n+1\rangle \qquad \text{if} \qquad 0 \le n \le N-2\ , \\ |N-1\rangle &\rightarrow |0\rangle\ . \end{aligned} \tag{3.1}$$

This model must be important in any ontological theory with determinism[4][5][7]: it is the simplest possible finite ingredient, returning to itself after N steps in time. Most models will use many such systems as a starting point, after which one may want to consider slight modifications of the primary evolution law, allowing the cogwheels somehow to interact. Many models may contain finite subsets that happen to be periodic; they will be described by the Cogwheel Model.

The reader may have noted that we use the Dirac ket notation to describe the states of the cogwheel. At this point, this was merely for convenience. However, there is a deeper reason for doing this. It may be useful to consider a *Hilbert space* associated to these states. For no other reason than to do some fancy mathematics at a later stage, we promote the states $|n\rangle$ to the status of basis elements of this (finite dimensional) Hilbert space. It certainly allows us to do statistics. We may decide to consider 'quantum states',

$$|\psi(t)\rangle = \sum_{n=0}^{N-1} \alpha_n(t)|n\rangle \ . \tag{3.2}$$

Suppose that these states evolve according to the following rule:

$$|\psi(t+1)\rangle = U\,|\psi(t)\rangle \ , \qquad U = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} \ , \tag{3.3}$$

then, if at $t = 0$ the coefficients α_n represent the probabilities that we are in state $|n\rangle$, that is, $W_n = |\alpha_n|^2$, this obviously will continue to be the case at all other times t (assuming t to take integral values). Again, we emphasize that quantum mechanics was only used as a notation; it is convenient to use Hilbert space to indicate how the probabilities evolve, but actually the evolution of this system, and its probabilities, are entirely classical.

Nevertheless, one may decide to proceed to a different basis. Since $U^N = \mathbb{I}$, the eigenvalues of U are $e^{-2\pi i k/N}$, $k = 0, \cdots, N-1$, so after diagonalization,

$$U = \begin{pmatrix} 1 & & & \\ & e^{-2\pi i/N} & & 0 \\ & & \ddots & \\ & 0 & & e^{-2\pi i(N-1)/N} \end{pmatrix} \ . \tag{3.4}$$

This, we can write as

$$U = e^{-iH} \ , \qquad H = \tfrac{2\pi}{N}\,\mathrm{diag}\,(0, 1, \cdots, N-1) \ . \tag{3.5}$$

The beat of the clock was used to define a unit of time. In its diagonal basis, the Cogwheel Model thus turns into a 'quantum mechanical' system whose 'Hamiltonian' has N equally spaced eigenvalues. This is a universal feature of all finite, periodic systems.

One may already conclude from this that quantum mechanics may well turn out to become a useful device for computing probabilities, even if a system is ontological and/or deterministic.

Different types of operators turn out to play a role. First, we have the *beables*. These are all observables that are diagonal in the original, 'ontological' basis. Beables simply refer to the state a system is in, multiplying it with a quantity that characterizes this state. For instance, one can project out a state: in the original basis,

$$P_n = |n\rangle\langle n| \tag{3.6}$$

is a beable. Or one can multiply with n:

$$\hat{n} = \sum_n n P_n \tag{3.7}$$

is also a beable.

Secondly, we have operators that change a state into another state. These will be referred to as *changeables*. For instance, U and H are changeables. Changeables can be introduced in *any* deterministic system. For example, in a classical description of the dynamics of our planetary system, the operator P_{ME} that interchanges the positions of Mars and the Earth, defined by

$$P_{ME}\,|\vec{x}_{\mathrm{Earth}},\,\vec{x}_{\mathrm{Mars}}\rangle = |\vec{x}_{\mathrm{Mars}},\,\vec{x}_{\mathrm{Earth}}\rangle\ , \tag{3.8}$$

is a totally legitimate operator of the 'changeable' kind. Although probably not very useful, it might be considered by planetary scientists in case they wish to investigate what would happen to the planetary system if the positions of Mars and the Earth (together with their moons) were simply interchanged. In planetary science, it would be even more strange to diagonalize this operator, although this would not at all be illegitimate.

The differences between classical theories and quantum mechanics concern the evolution laws. In quantum mechanics, it appears that a beable can evolve to become a changeable, and *vice versa*. This does not happen in Newtonian mechanics; there, beables will always be beables, and changeables evolve into changeables.

In quantum mechanics, the spin of a particle can rotate, for instance when a magnetic field is applied to it. Under the influence of a field in the y-direction, the operator σ_z, which is a beable when measuring the spin in the z-direction, then may rotate to become the operator σ_x, which is known to be a changeable when viewed in the basis of σ_z: it adds or subtracts one unit of spin in the z-direction. This sets quantum mechanics apart from the other theories, but is this difference fundamental?

Beables evolving into changeables and back, this is the phenomenon that we should focus on. If any of the spin components, σ_x, σ_y, or σ_z would have an ontological interpretation, then quantum mechanics would not be a deterministic theory, since magnetic fields can transform these operators into things without an ontological interpretation – changeables. However, could it be that all of these operators, σ_x, σ_y, and σ_z, are actually complicated functions of beables and changeables of some underlying theory? Could these complicated functions involve, or even interfere with, the 'free will' of an observer? We claim that, if, in an underlying theory, beables only evolve into beables, then in an *effective* quantum theory, σ_x, σ_y, and σ_z may well be able to evolve into one another, under given circumstances.

There is a continuum counterpart to the Cogwheel Model. Let a variable $q(t)$ take values on the circle. Let it rotate uniformly at angular velocity one:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = 1\ . \tag{3.9}$$

Defining an "ontological basis" of states $|q\rangle$, and in that Hilbert space the displacement operator $\hat{p} = -i\partial/\partial q$, we find

$$\dot{q} = -i[q,H]\ , \qquad H = \hat{p}\ , \tag{3.10}$$

and since all states are periodic in time with period 2π , we have

$$e^{-2\pi H}|\psi\rangle = |\psi\rangle \ , \qquad e^{2\pi H} = 1 \ , \qquad H = n = 0, \pm 1, \pm 2, \ \cdots . \tag{3.11}$$

This is a special case of the system with N continuous variables q_i , $i = 1, \cdots, N$, obeying some equation of motion

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = f_i(\vec{q}) = -i[q_i, H] \ , \tag{3.12}$$

for which we can use the Hamiltonian

$$H(\vec{p}, \vec{q}) = p_i f_i(\vec{q}) + g_i(\vec{q}) \ . \tag{3.13}$$

Note that, even though we used a fully quantum mechanical description and notation, our system is still totally deterministic. The Hamiltonians (3.11) and (3.13) may resemble quantum mechanical Hmiltonians, but they differ from those in one very important sense: *there is no lowest energy state (ground state).* There appears to be a close relation between periodic deterministic systems and quantum harmonic oscillators, if a good explanation can be found why the negative energy eigenstates in Eq. priodicstates are projected out, even if interactions are introduced.

4. Quantum statics *vs* quantum dynamics

There is an other way in which the classical systems described in the previous section differ from quantum mechanical ones, in spite of the notation used: in quantum mechanics, it *appears to be possible quite generally to devise experiments that produce states that are eigenstates of whatever hermitean operator one may imagine.* This is quite remarkable, and not at all true for planets. In any 'hidden variable' theory for quantum mechanics, this should be one of the most urgent issues that has to be addressed: how could this feature be possible? How do we explain this? Why are the beables so thoroughly mixed with the changeables?

We suspect that this phenomenon must be the result of a 'renormalization group transformation' from the Planck scale to the Standard Model scale (see Section 7. This transformation probably causes a complete mixture of the two sets of operators. But we suspect that there is more. Why does a transition from a microscopic description of Boltzmann molecules to a macroscopic description of a Van der Waals gas not appear to mix beables with changeables? This is the kind of questions we have to face. We insist that this question is one involving the *dynamical equations* describing the time evolution in quantum mechanics. When we send a beam of electrons through a Stern-Gerlach arangement, we can easily choose the axis along which the spin is diagonalized, thus mixing beables with changeables all the time. Any time a neutral pion decays into two photons, the two photons appear in an entangled state (since the total spin is zero), and the number of experiments proposed, and actually done, is seemingly endless.

This explains why, very often, in discussions of some quantum weirdness involving entanglement, the description of the procedure needed to generate the states in question, is hardly given any thought (such as in Conway and Kochen's paper[3]). It is taken for granted that the states can be produced. The rest of the discussion then involves the static state only. For our present discussion, however, we think it is of tantamount importance to include the question of dynamics. Eigenstates of one kind of operator appear to evolve into eigenstates of other operators that do not commute. According to the theory advanced here, this must be an illusion. The operators that should be used at the Planckian level should all be beables. Can we realize this in a theory? How can such a theory generate evolution laws that appear to mix beables and changeables incessantly?

5. The Cellular Automaton

In this section, a model will be described that at first sight might seem to do exactly what we want: there are only beables evolving into beables at a microscopic level, whereas something resembling a full-fledged quantum field theory emerges at large scales. The caveats will come at the end.

We consider a *cellular automaton.* The construction is such that the same procedure as the one used for the Cogwheel model can be applied here; in this case, the model will be time-reversible.

Space and time are both taken to be discrete: we have a D dimensional spacelike lattice, where positions are indicated by integers: $\vec{x} = (x^1,\, x^2,\, \cdots,\, x^D)\,,\ x^i \in \mathbb{Z}\,.$ Also time t will be indicated by integers, and time evolution takes place stepwise. The physical variables $F(\vec{x},\, t)$ in the model could be assumed to take a variety of forms, but the most convenient choice is to take these to be integers modulo some number $\mathbb{N}$. Most importantly, these physical degrees of freedom are defined to be attached only to the even sites:

$$\sum_{i=1}^{D} x^i + t \ = \ \text{even}. \tag{5.1}$$

Furthermore, the data can be chosen freely at two consecutive times, so for instance at $t = 0$, we can choose the initial data to be $\{F(\vec{x},\, t),\ F(\vec{x},\, t+1)\}$.

The dynamical equations of the model can be chosen in several ways, provided that they are time reversible. To be explicit, we choose them to be as follows:

$$\begin{aligned} &F(\vec{x},\, t+1) \ = \ F(\vec{x},\, t-1) + \\ &Q\,\big(\, F(x^1 \pm 1, x^2, \cdots, x^D,\, t),\, \cdots,\, F(x^1, \cdots, x^D \pm 1,\, t)\,\big) \ \text{Mod}\ \mathbb{N}\,, \\ &\text{when} \quad \textstyle\sum_i x^i + t \quad \text{is odd}\,, \end{aligned} \tag{5.2}$$

where the integer Q is some arbitrary given function of all variables indicated: all nearest neighbors of the site $\vec{x}$ at time t. This is time reversible because we can find $F(\vec{x},\, t-1)$ back from $F(\vec{x},\, t+1)$ and the neighbors at time t. Assuming Q to be a sufficiently irregular function, one generally obtains quite non-trivial cellular automata this way. Models of this category are often considered in computer animations.

We now discuss the mathematics of this model using Hilbert space notation[6]. We switch from the Heisenberg picture, where states are fixed, but operators such as the beables $F(\vec{x},\,t)$ are time dependent, to the Schrödinger picture. Here, we call the operators F on the even sites $X(\vec{x})$, and the ones on the odd sites $Y(\vec{x})$. As a function of time t, we alternatingly update $X(\vec{x})$ and $Y(\vec{x})$, so that we construct the evolution operator over two time steps. Keeping the time parameter t even:

$$U(t,\,t-2)=A\cdot B\ , \tag{5.3}$$

where A updates the data $X(\vec{x})$ and B updates the data $Y(\vec{x})$.

Updating the even sites only, is an operation that consists of many parts, each defined on an even coordinate $\vec{x}$, and all commuting with one another:

$$A=\prod_{\vec{x}\ \mathrm{even}}A(\vec{x})\ ,\quad [A(\vec{x}),\,A(\vec{x}\,')]=0\ , \tag{5.4}$$

whereas the B operator refers only to the odd sites,

$$B=\prod_{\vec{x}\ \mathrm{odd}}B(\vec{x})\ ,\quad [B(\vec{x}),\,B(\vec{x}\,')]=0\ . \tag{5.5}$$

Note now, that the operators $A(\vec{x})$ and $B(\vec{x}\,')$ do not all commute. If $\vec{x}$ and $\vec{x}\,'$ are neighbors, then

$$\vec{x}-\vec{x}\,'=\vec{e},\ |\vec{e}|=1\ \rightarrow\ [A(\vec{x}),\,B(\vec{x}\,')]\neq 0\ . \tag{5.6}$$

It is important to observe here that both the operators $A(\vec{x})$ and $B(\vec{x})$ only act in finite subspaces of Hilbert space, so we can easily write them as follows:

$$A(\vec{x})=e^{-i\mathsf{a}(\vec{x})}\ ,\qquad B(\vec{x})=e^{-i\mathsf{b}(\vec{x})}\ . \tag{5.7}$$

We can write

$$\mathsf{a}(\vec{x})=-\mathcal{P}_x(\vec{x})\ Q(\{Y\})\ ,\quad \mathsf{b}(\vec{x})=-\mathcal{P}_y(\vec{x})\ Q(\{X\})\ , \tag{5.8}$$

where $\mathcal{P}_x(\vec{x})$ is the generator for a one-step displacement of $X(\vec{x})$ in its internal space modulo $\mathbb{N}$:

$$e^{i\mathcal{P}_x(\vec{x})}|X(\vec{x})\rangle\stackrel{\mathrm{def}}{=}|X(\vec{x})+1\ \mathrm{Mod}\ \mathbb{N}\rangle\ , \tag{5.9}$$

and, similarly, $\mathcal{P}_y(\vec{x})$ generates one step displacement of the function $Y(\vec{x})$. We see that

$$[\mathsf{a}(\vec{x}),\,\mathsf{a}(\vec{x}\,')]=0\ ,\quad [\mathsf{b}(\vec{x}),\,\mathsf{b}(\vec{x}\,')]=0\ ,\quad \forall\,(\vec{x},\,\vec{x}\,')\ ; \tag{5.10}$$

$$[\mathsf{a}(\vec{x}),\,\mathsf{b}(\vec{x}\,')]=0\quad \text{only if}\quad |\vec{x}-\vec{x}\,'|>1\ . \tag{5.11}$$

A consequence of Eqs. (5.10) is that also the products A in Eq. (5.4) and B in Eq. (5.5) can be written as

$$A=e^{-i\sum_{\vec x\ \mathrm{even}}\mathsf{a}(\vec x)}\ ,\qquad B=e^{-i\sum_{\vec x\ \mathrm{odd}}\mathsf{b}(\vec x)}\ . \tag{5.12}$$

However, now A and B do not commute. Nevertheless, we wish to compute the total evolution operator U for two consecutive time steps, writing it as

$$U=A\cdot B=e^{-i\mathsf{a}}\,e^{-i\mathsf{b}}=e^{-2iH}\ . \tag{5.13}$$

For this calculation, we could use the power expansion given by the Campbell-Baker-Haussdorff formula,

$$\begin{aligned}&e^P\,e^Q=e^R\ ,\\ &R=P+Q+\tfrac12[P,Q]+\tfrac1{12}[P,[P,Q]]+\tfrac1{12}[[P,Q],Q]+\tfrac1{24}[[P,[P,Q]],Q]+\cdots\ ,\end{aligned} \tag{5.14}$$

a series that continues exclusively with commutators. Replacing P by $-i\mathsf{a}$, Q by $-i\mathsf{b}$ and R by $-2iH$, we find a series for the 'Hamiltonian' H in the form of an infinite sequence of commutators. Now note that the commutators between the local operators $\mathsf{a}(\vec x)$ and $\mathsf{b}(\vec x\,')$ are non-vanishing only if $\vec x$ and $\vec x\,'$ are neighbors, $|\vec x-\vec x\,'|=1$. Consequently, if we insert the sums (5.12) into Eq. (5.14), we obtain again a sum:

$$\begin{aligned}&H=\sum_{\vec x}\mathcal{H}(\vec x)\ ,\\ &\mathcal{H}(\vec x)=\tfrac12\mathsf{a}(\vec x)+\tfrac12\mathsf{b}(\vec x)+\mathcal{H}_2(\vec x)+\mathcal{H}_3(\vec x)+\cdots\ ,\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}\mathcal{H}_2(\vec x)&=-i\tfrac14\sum_{\vec y}[\mathsf{a}(\vec x),\,\mathsf{b}(\vec y)]\ ,\\ \mathcal{H}_3(\vec x)&=-\tfrac1{24}\sum_{\vec y_1,\vec y_2}[\mathsf{a}(\vec x)-\mathsf{b}(\vec x)\,,\ [\mathsf{a}(\vec y_1),\mathsf{b}(\vec y_2)]]\ ,\quad\text{etc.}\end{aligned} \tag{5.16}$$

All these commutators are only non-vanishing if the coordinates $\vec y$, $\vec y_1$, $\vec y_2$, etc., are all neighbors of the coordinate $\vec x$. It is true that, in the higher order terms, next-to-nearest neighbors may enter, but still, one may observe that these operators are all local functions of the 'fields' $F(\vec x,\,t)$, and thus we arrive at a Hamiltonian H that can be regarded as the sum over D-dimensional space of a Hamilton density $\mathcal{H}(\vec x)$, which has the property that

$$[\mathcal{H}(\vec x),\,\mathcal{H}(\vec x\,')]=0\ ,\quad\text{if}\quad|\vec x,\,\vec x\,'|\gg1\ . \tag{5.17}$$

At every finite order of the series, the Hamilton density $\mathcal{H}(\vec x)$ is a finite-dimensional Hermitean matrix, and therefore, it will have a lowest eigenvalue h. In a large but finite volume V, the total Hamiltonian H will therefore also have a lowest eigenvalue, obeying

$$E_0>h\,V\ . \tag{5.18}$$

The associated eigenstate $|0\rangle$ might be identified with the 'vacuum'. This vacuum is stationary, even if the automaton itself may have no stationary solution. The next-to-lowest eigenstate may be a one-particle state. In a Heisenberg picture, the fields $F(\vec{x},t)$ may create a one-particle state out of the vacuum. Thus, we arrive at something that resembles a genuine quantum field theory. The states are quantum states in complete accordance with a Copenhagen interpretation. The fields $\mathsf{a}(\vec{x},t)$ and $\mathsf{b}(\vec{x},t)$ should obey the Wightman axioms.

There are three ways, however, in which this theory differs from conventional quantum field theories. One is, of course, that space and time are discrete. Well, maybe there is an interesting 'continuum limit', in which the particle mass(es) is(are) considerably smaller than the inverse of the time quantum. Secondly, no attempt has been made to arrive at Lorentz invariance. Thus, the dispersion relation of these particles, if they obey any at all, may be nothing resembling conventional physical particles.

But the third difference is more profound. It was tacitly assumed that the Campbell-Baker-Haussdorff formula converges. This is usually not the case. One can argue that the series will converge well only if sandwiched between two eigenstates $|E_1\rangle$ and $|E_2\rangle$ of H, where E_1 and E_2 are the eigenvalues, that obey

$$2|E_1 - E_2| < 2\pi \ , \tag{5.19}$$

where the first factor 2 is the one in Eq. (5.13).

This is a severe restriction, but perhaps one can argue that 2π here is the Planck energy, and in practice, when we do quantum mechanics, we only look at energies, or rather energy differences, that indeed are much smaller than the Planck energy. But does this mean that transitions with larger energy differences do not occur? We must realize that energy is not exactly conserved in this model. Since time is discrete, energy is only conserved modulo π, and this may indicate that our 'vacuum state' is not stable after all. The energy can jump towards other states by integer multiples of π. Thus, our conclusions must be drawn with considerable caution.

The conclusion we do wish to draw is that procedures borrowed from genuine quantum mechanics can be considered, and they may lead to a rearrangement of the states in such a way that beables and changeables naturally mix, leaving an effective description of a system for which only quantum mechanical language applies. This, we think, is all we really need to understand why it is quantum mechanics that seems to dominate the world of the small things.

6. Quantum symmetries

One reason why the quantum mechanical method, used as a tool, may show to be extremely powerful, is that it might exhibit symmetries that would be obscure otherwise. We here discuss a simple example. Consider a 1+1 dimensional cellular automaton of the kind introduced in the previous section (it could be easily extended to more spacial dimensions). The data, here called $\sigma_{x,t}$, $x+t$ even, now only takes the values ± 1, and the evolution rule is chosen to be a very simple one:

$$\sigma_{x+1,t+1} = \sigma_{x,t}\,\sigma_{x+2,t}\,\sigma_{x+1,t-1}\ . \tag{6.1}$$

Classically, this has a displacement symmetry when

$$\begin{pmatrix} x \\ t \end{pmatrix} \rightarrow \begin{pmatrix} x+\delta x \\ t+\delta t \end{pmatrix}\ ; \qquad \delta x + \delta t \text{ even}\ . \tag{6.2}$$

Now, however, introduce an operator at all *odd* sites, to be called $\sigma^1_{x,t}$, $x+t$ odd. This operator is defined such that, in a Heisenberg picture, it switches the sign of the σ data at the same x at time $t+1$ and at time $t-1$. At spacelike separations from (x,t), no other changes take place. We see that this operation leaves the evolution equation intact. It is useful to write the σ data at the original, even sites as $\sigma^3_{x,t}$. Writing them as Pauli matrices, we have

$$\sigma^3_{x,t\pm 1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \qquad \sigma^1_{x,t} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ . \tag{6.3}$$

Then observe that, applying four such operations may leave the state unchanged:

$$\sigma^1_{x,t+1} = \sigma^1_{x-1,t}\,\sigma^1_{x+1,t}\,\sigma^1_{x,t-1}\ , \tag{6.4}$$

which is the same equation as the original evolution equation! Thus, a displacement with $\delta x + \delta t$ odd, commutes with the Hamiltonian, provided it is combined with the unitary operator

$$U = \prod_{\substack{x \text{ even} \\ \text{and odd}}} \tfrac{1}{\sqrt{2}}(\sigma^1(x) + \sigma^3(x))\ , \tag{6.5}$$

which switches σ^1 with σ^3. This way, we managed to extend the symmetry (6.2) to hold also for odd displacements. This would have made no sense in the classical description. We see, for instance, that now energy and momenta will be well defined in a larger Brillouin zone than what one would have expected classically.

In some cases, there is a simpler way to extend the symmetry group of a system by using the quantum notation. Consider a one dimensional array of data, and introduce the displacement operator U for the displacement by one unit:

$$U|\{f_x\}\rangle = |\{f_{x-1}\}\rangle\ ; \qquad \hat{x}U = U(\hat{x}+1)\ . \tag{6.6}$$

Its eigenstates $|p,r\rangle$ have eigenvalues with norm one:

$$U|p,r\rangle = e^{ip}|p,r\rangle\ ; \qquad -\pi \le p < \pi\ . \tag{6.7}$$

A fractional displacement can then also be defined:

$$U(a) \stackrel{\text{def}}{=} e^{iap}\ , \qquad a \text{ real}\ . \tag{6.8}$$

7. The Renormalization Group

Statistical analysis will be inevitable if one has a model whose ontological degrees of freedom are defined at a scale such as the Planck scale, while we want to study physics at the Standard Model scale, more than 16 orders of magnitude onwards. One might have thought that it would be a miracle if, at such large scale separations, any significant type of structure survives; rather, one would have expected completely homogeneous white noise. This might be the reason for using quantum statistics. Once a cellular automaton has been turned into a quantum field theory, as in Section 5, it can be subjected to *scaling* across huge orders of magnitude using the *renormalization group*. Imagine that the quantum field theory in question can be approximately described using perturbation theory. Many of its coupling strengths would run to zero (these are the so-called *marginal* couplings). The renormalizable couplings would scale logarithmically, while the super renormalizable ones (such as the mass terms) become more significant at larger distances; they must be tuned to be small at the initial scale. Since the renormalization group tends to cause all quantum operators to mix with one another, one can imagine that this procedure causes a thorough mixing between beables and changeables, which could explain the confusion physicists encounter today when studying quantum effects.

In a theory defined on a lattice, the complete space for all momentum values has Bloch domain walls, the cell walls of the dual lattice. This, of course, violates rotation invariance. The renormalization group, however, requires that one concentrates on only the very tiny values of the momenta, so that the domain walls will run out of sight, and rotational symmetry is recovered. This is a well-known feature in lattice field theories, where one notices that the lattice artifacts go away as one goes to larger distance scales.

8. Dissipation or loss of information

Non-trivial structures are thus expected to arise at time scales, and at distance scales, that are very large compared to the lattice scale (referred to as 'Planck scale' in the sequel). Now, experts in cellular automata may jump on their feet saying that, of course, large scale structures are commonplace in cellular automata, but then one forgets something. The automata discussed here are time-reversible, and we consider generic initial states. Such automata in general produce completely random, white noise; long range structures only arise if there is a considerable amount of dissipation. In physical terms: the entropy of the resulting states should be not too high. A typical example is Conway's 'game of life'. This is an automaton that rapidly converges towards states that are close to the empty states: states with much more zeros than ones, showing only short term local periodicities. Such models cannot be time-reversible. It is this author's experience that cellular automata that are time-reversible and start off with a generic initial state, always lead to white noise.

There is a good reason generally to expect time reversible cellular automata to go onto a white noise mode, once a sufficiently generic evolution law and a sufficiently generic initial state is chosen. In that case, the recurrence time will diverge rapidly as the volume is chosen to be large, and a sizeable fraction of all possible configurations will be reached before the initial state is reobtained. All those states will have exactly equal probability, so that one cannot expect any large scale structure to develop.

To realize more structure at long distances, one needs to modify the rules for an automaton: there must be dissipation. This means that so much information is erased during the evolution, that any generic state quickly evolves to become a member of a very small subset of states, each showing structure at very large distance and time scales. Structure at large scales can only arise if many states exist whose probabilities are different; these probabilities can only be different if the number of different past states that evolve into the same final state can vary considerably.

The prototype of a model with information loss is the *Defect Cogwheel Model.* A cogwheel with 4 teeth evolves, but when it is in the next-to last position, it advances two steps instead of one. The evolution law is therefore:

$$\text{At each time step:} \quad \begin{array}{ccc} 0 & \rightarrow & 1\,, \\ 1 & \rightarrow & 2\,, \\ 2 & \rightarrow & 0\,, \\ 3 & \rightarrow & 0\,. \end{array} \tag{8.1}$$

However, if we would identify the four states of this cogwheel with basis elements of a 'Hilbert space', the evolution operator $U(t, t+1)$ would not be unitary:

$$U(t, t+1) \stackrel{?}{=} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{8.2}$$

Clearly, this is not the way to apply the techniques from quantum mechanics. What can be done, is to associate basis elements of Hilbert states not to single ontological states, but to *equivalence classes* of states. Two states are in the same equivalence class iff there exists a time interval T such that at $t \ge T$ they evolve into the same final state.

Equivalence classes defined this way show much resemblance to gauge equivalence classes in gauge theories. We will return to this observation.

The Defect Cogwheel Model has three equivalence classes, and in terms of these basis elements the evolution matrix is

$$U(t, t+1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} . \tag{8.3}$$

One might wonder why the fourth row and column should be considered at all, in the defect cogwheel model of Eq. (8.1). The point is that, in much more complex cases, it is impossible to distinguish those states that have no distance past at all from the ones with a large number of distinct pasts.

It is important to note that, in the absence of information loss, or dissipation, conventional classical theories such as Newtonian mechanics, show a feature that might cause some trouble, in particular when one wishes to apply the 'quantum method' advertised here: the usual classical mechanics models are not entirely deterministic. We simply remind the reader that initial states must be defined with infinite precision, that is, an infinite sequence of decimal places, if one wishes to keep its future evolution under control. A prototype model is the following evolution law:

$$\begin{array}{lcl} t=0 & : & x=1.23456789012345\cdots \\ t=1 & : & x=2.41638507294163\cdots \\ t=2 & : & x=4.62810325476981\cdots \\ & & \text{etc.} \end{array} \tag{8.4}$$

Thus, the decimal places at odd sites move two spaces forwards, and the decimal places at even sites move two steps to the back.

What this means is that, insignificant decimals soon turn into more significant ones and vice versa. Eq. (8.4) was merely put forward to indicate how the evolution forwards as well as backwards can be fully 1 to 1, while nevertheless prediction of the values at large time becomes horrendously difficult, and, in a sense, even impossible. With information loss, we can have a model that, when followed forward in time, is completely deterministic *and* predictable:

$$\begin{array}{lcl} t=0 & : & x=1.23456789012345\cdots \\ t=1 & : & x=1.02345678901234\cdots \\ t=2 & : & x=1.00234567890123\cdots \\ & & \text{etc.} \end{array} \tag{8.5}$$

In elementary particle physics, there are some indications that information may get lost at the Planck scale. We know that black holes are solutions to Einstein's equations of General Relativity. At the Planck scale, gravity becomes a strong force; non-perturbative effects are inevitable, and that means that black holes must play a role as intermediate states. Classically, objects can fall in without leaving any information behind. It is true that *quantum theory*, when applied to black holes, forces particles to come out again, in the form of Hawking radiation. These particles can redeliver the lost information in a quantum-entangled form. In our theory, quantum states are equivalence classes of information, so that, indeed, a quantum theory preserves information in the quantum mechanical sense.

The price paid is then a more subtle one. It is known as the *holographic principle*[9]. This says that the *total number of independent quantum states* can be enumerated on a surface at the boundary of a system rather than in the bulk of a system. This we interpret by deducing that information residing in the bulk will be lost; it is linked to information at the surface via the informational equivalence classes. This, we suspect, is what 'holography' really is about: it tells us that the information that can be retrieved from a system is limited.

An other consequence of information loss is that systems that are described by continuous degrees of freedom obeying differential equations in time, such as Eq. (3.12), will tend to convert towards quantized orbits. This is explained in Fig. 1. If the evolution

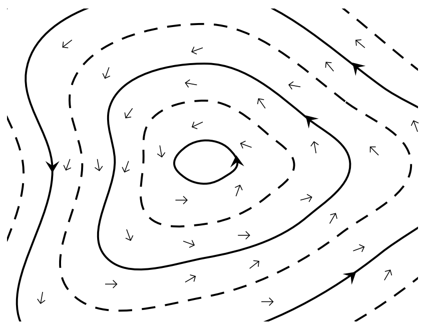

Figure 1: The emergence of quantum states when there is information loss. Solid lines: stable attractors. Dashed lines: unstable closed orbits, acting as attractors under time reversal.

equation is not volume preserving, it will generate stable attractors, that could be represented by quantum numbers. Thus, with such starting points it might be easier to arrive at descriptions where deterministic systems develop quantum behavior.

9. The Determinant Model

A simple model can be used to display how a continuous deterministic system with information loss can turn into a quantum system. Let there be given a hermitean $N\times N$ matrix H . Assume that we have two continuous degrees of freedom, and angle $\phi\in[0,\,2\pi)$, and a real number ω . Assume

$$\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = \omega(t) \ , \tag{9.1}$$

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = -\kappa\, f(\omega)\, f\,'(\omega) \ , \qquad f(\omega)=\det(H-\omega) \ , \tag{9.2}$$

where $f\,'$ is the derivative of f . One quickly ascertains that the zeros of $f(\omega)$ are stable attractors of the system. The zeros of $f\,'$ are unstable. Thus, the informational equivalence classes are numbered by the zeros of f , which, due to Eq. (9.2), are the eigenvalues of H . So, each eigenvalue $\omega_n,\ n=1,\cdots,N$, corresponds to a state $|\psi_n\rangle$. In the limit $t\to\infty$, we have periodicity with period $T=2\pi/\omega_n$, so, the wave function can be written as

$$\psi=e^{ik(\phi-\omega t)}|\psi_n\rangle \ , \qquad k\in\mathbb{Z} \ . \tag{9.3}$$

There are reasons to suspect that the value $k=1$ has to be singled out.[8] This is the one state that has the right periodicity $2\pi/\omega$; higher k values would correspond to equivalence classes that have higher periodicity, because there is information loss that removes the distinction between points on the orbit and other points on the same orbit. With this extra connotation, we find that Hilbert space consists of the quantum states

$$|\psi(t)\rangle=\sum_{n=1}^{N}\alpha_n|\psi_n\rangle e^{-i\omega t} \ , \tag{9.4}$$

which of course is the set of solutions of the quantum system whose Hamiltonian is H.

10. Discussion

Our work can serve as a special approach towards interpreting quantum mechanics. We assume that there are underlying deterministic equations, but today's physics may not yet have reached the stage that we can speculate about Nature's true ontological degrees of freedom in a meaningful way. Instead, we can work with the quantum equations the way we have been taught in the text books without being mesmerized by the strange picture usually sketched about 'reality'. Reality is described by degrees of freedom that we have not yet understood, which is why we think we perceive phenomena such as interference effects.

Yet we hope that this work is more than just that. It could also serve to put us on track in our search for the 'true equations' that define the dynamics of this world at the Planck scale. Superstring theory is usually presented without any attempt to look beyond its Hilbert space formulation. At the same time, while indeed suggestive and promising, the true interpretation of string theory, or its more sophisticated successor M-theory, is mysterious to such an extent that further insights are barred by lack of intuition. A deterministic continuation of string theory would shed a more understandable light on features such as black holes and holography, and would also help us understand cosmology, or cosmogeny, in a more satisfactory manner.

One could add that the two-dimensional nature of the string world sheet, as well as the fact that its primary degrees of freedom are harmonic oscillators, make string theory a particularly interesting candidate for a reformulation in terms of something deterministic.

An interesting speculation is that *loxal gauge symmetries* are due to information loss. The gauge equivalence classes are then identified with our informational equivalence classes: the 'information' as to which element of the gauge equivalence class we are in is not being retained by the system. A next step could then be that this may also hold for the coordinate equivalence classes of General Relativity. *There are coordinates, but we cannot keep track of them.* This might have some bearing on the cosmological constant problem.

More extensive discussions can be found elsewhere[4][8] and are in the making.

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