Nuclear Physics B62 (1973) 444–460. North-Holland Publishing Company

AN ALGORITHM FOR THE POLES AT DIMENSION FOUR IN THE DIMENSIONAL REGULARIZATION PROCEDURE

G. 't HOOFT *
CERN, Geneva

Received 25 June 1973

Abstract: One-loop Feynman diagrams, when regularized using the continuous dimension method, exhibit single poles at $n = 4$, and these poles can be eliminated by a counterterm $\Delta \mathcal{L}$ in the Lagrangian. In this paper a simple algorithm is derived to express $\Delta \mathcal{L}$ in terms of the components of the Lagrangian $\mathcal{L}$ without performing integrations. It can be used in dimensional regularization, but also to derive Callan-Symanzik type equations for small distance behaviour in any renormalizable theory. The result of a calculation of the divergencies in quantum gravity is reported.

1. Introduction

The continuous dimension method of regularization has turned out to be extremely useful in theories with local symmetries, since all these symmetries are left intact [1, 2]. After application of this regularization method an infinity remains: poles in the n plane at $n = 4$. Only after removal of these poles by means of a renormalization counterterm $\Delta \mathcal{L}$ in the Lagrangian, the limit $n \to 4$ can be taken.

It is of importance that the new Lagrangian $\mathcal{L} + \Delta \mathcal{L}$ has the same symmetry structure as the old one, $\mathcal{L}$, so that we can repeat the procedure order by order in perturbation theory. In fact, investigation of the symmetry properties of $\mathcal{L} + \Delta \mathcal{L}$ has been performed in order to check the self-consistency of the method (see sect. 7 of ref. [3]).

In this paper we derive an algorithm for a fast calculation of the counterterms $\Delta \mathcal{L}$ for all one-loop diagrams. Its applications are probably diverse. It will not only simplify infinity calculations in complicated gauge theories and the theory of quantum gravity. Knowledge of $\Delta \mathcal{L}$ also enables one to compute small distance limits of Green functions, as we have shown in a previous publication [4]. For instance, using our algorithm it will be quite easy to deduce that certain gauge theories have a smooth and calculable small distance behaviour, contrary to other re-

* On leave from the University of Utrecht, Utrecht.

normalizable theories, like ϕ^4 or Q.E.D. This phenomenon has also been observed in ref. [5]. Furthermore, calculable conformally invariant theories ($\Delta\mathcal{L} = 0$) can be found (see sect. 8).

It will be shown that $\Delta\mathcal{L}$ can be expressed in terms of the "second derivative" of the Lagrangian $\mathcal{L}$ with respect to the fields in $\mathcal{L}$. It is convenient to use a formulation in terms of background fields [6, 7], which we explain in sect. 2. In this paper we show how the residues for all one-loop diagrams can be calculated in an elegant way by exploiting the invariance of the problem under certain local gauge transformations of the background fields. The obtained expression has been checked by explicit calculation of all different types of diagrams. Our result, which can be used as a starting point for calculations in any renormalizable theory, is given by formula (4.15), or, if fermions are present, formula (6.12).

Furthermore, one may insert the classical equation of motion into our expressions for $\Delta\mathcal{L}$. This is motivated in sect. 7. In certain examples we show how to use our algorithm (sect. 8). Here we see that it reduces the rather lengthy calculation of the infinities in some gauge theories to a few lines (provided a convenient gauge is chosen).

Finally, one can now study quantum gravity from this point of view. We postpone the details of our gravity calculations to a future publication [8], but we anticipate on our result in sect. 9.

2. The background field method *

Let us consider first a field theory with real ** Bose fields $A_i(x)$, where i is any kind of index, including possibly a Lorentz index. Let it be described by a Lagrangian

$$\mathcal{L}[A, x] = \mathcal{L}[A_i(x), \partial_\mu A_i(x)] \ .$$

We shall now define what we call the first derivative $\mathcal{L}'$, and the second derivative (W, N, M), of this Lagrangian with respect to the fields A_i:

$$\mathcal{L}[A+\phi, x] = \mathcal{L}[A, x] + \phi_i(x)\, \mathcal{L}'_i[A, x] + \tfrac{1}{2}\, \partial_\mu\phi_i(x)\, W^{\mu\nu}_{ij}[A, x]\, \partial_\nu\phi_j(x)$$
$$+ \phi_i N^\mu_{ij}[A, x]\, \partial_\mu \phi_j + \tfrac{1}{2}\, \phi_i M_{ij}[A, x]\, \phi_j + \mathrm{O}(\phi^3) + \text{total space-time derivative}\,. \tag{2.1}$$

[As an example, in ϕ^4 theory, we would have: $\mathcal{L}' = (\partial^2 - m^2)A - \frac{1}{6}\lambda A^3$; $W^{\mu\nu} = -\delta^{\mu\nu}$; $N^\mu = 0$; $M = -m^2 - \frac{1}{2}\lambda A^2$.]

In general, the classical equations of motion are given by

$$\mathcal{L}'_i[A, x] = 0\,. \tag{2.2}$$

* Refs. [6, 7].

** Time components of real vector fields are of course purely imaginary.

The propagators of the theory are usually obtained by considering W and M with the field A replaced by zero: they are the inverse of the matrix

$$-W^{\mu\nu}_{ij}[0]\, k_\mu k_\nu - M_{ij}[0] \,.$$

The definition (2.1) will enable us to abbreviate the expressions for one-loop diagrams. Consider namely an irreducible one-loop diagram. All external and internal lines correspond to some of the fields A_i. The vertices are described by the trilinear and quadrilinear terms in $\mathcal{L}[A]$. If, however, we call the internal lines ϕ lines, then the same vertices are delivered by $W[A] - W[0], N[A]$, and $M[A] - M[0]$, as one can easily convince oneself. Hence, all one-loop diagrams are also generated by the Lagrangian

$$\mathcal{L}[\phi, x] = \tfrac{1}{2}\, \partial_\mu \phi_i W^{\mu\nu}_{ij}[A,x]\, \partial_\nu \phi_j + \phi_i N^\mu_{ij}[A,x]\, \partial_\mu \phi_j + \tfrac{1}{2}\, \phi_i M_{ij}[A,x]\, \phi_j \,. \tag{2.3}$$

The fields A_i are now external fields, and W, N and M in eq. (2.3) must be considered as external space-time dependent source functions *. Only the fields ϕ_i in eq. (2.3) are quantized.

By adding total space-time derivatives to $\mathcal{L}[\phi, x]$, one can always arrange things such that

$$W^{\mu\nu}_{ij} = W^{\mu\nu}_{ji} = W^{\nu\mu}_{ij} \,,$$
$$N^\mu_{ij} = -N^\mu_{ji} \,, \quad M_{ij} = M_{ji} \,. \tag{2.4}$$

In the following sections we assume (2.4) to be valid. The case of fermions will be considered in sect. 6.

3. Some pole terms, and the algorithm for scalar fields

Things simplify considerably if we make the restriction

$$W^{\mu\nu}_{ij} = -\delta^{\mu\nu}\delta_{ij} \,. \tag{3.1}$$

This holds for all ordinary renormalizable field theories, including gauge theories provided one chooses a Feynman-like gauge. The propagator is then

$$\delta_{ij}(k^2 - M[0] - i\epsilon)^{-1} \tag{3.2}$$

This we expand in a finite series:

$$\frac{\delta_{ij}}{k^2 - i\epsilon}\left[1 + \frac{M[0]}{k^2 - i\epsilon} + \left(\frac{M[0]}{k^2 - i\epsilon}\right)^2 + \frac{M^3[0]}{(k^2 - i\epsilon)^2(k^2 - M[0] - i\epsilon)}\right] \,. \tag{3.3}$$

* In all theories that are renormalizable by power counting, W is space-time independent. Only in the theory of gravity the more general case is to be considered [8].

Integrals that contain the last term of this expression will never diverge, so that this term will never contribute to a pole in the dimension plane at $n = 4$. To obtain these poles, therefore, we merely need to consider the first few terms of the expansion, which correspond exactly to inserting the $M[0]$ term as a two-point vertex in a diagram with zero mass particles.

So, from now on, we treat the complete $M[A]$ term as an external vertex, replacing the propagator (3.2) by

$$\delta_{ij}/(k^2 - i\epsilon) \,. \tag{3.4}$$

As we shall see, this simplifies our pole calculations.

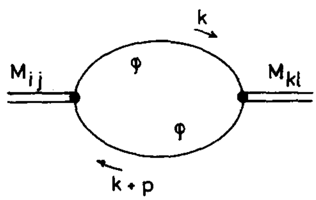

Fig. 1.

Consider now the diagram of fig. 1. The external double lines are just an abbreviation for the combination of A lines that make up M. The vertices are given by the last term in (2.3). The amplitude is

$$\tfrac{1}{4}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\frac{1}{(2\pi)^4 i}\int \mathrm{d}^n k\,\frac{1}{(k^2 - i\epsilon)((k+p)^2 - i\epsilon)} \,. \tag{3.5}$$

Here n denotes the number of space-time dimensions. For non-integer n this integral is equal to [1]

$$\tfrac{1}{4}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\frac{\pi^{\frac{1}{2}n-4}\,\Gamma(2-\frac{1}{2}n)}{16}\int_0^1 \mathrm{d}x\,[x(1-x)p^2]^{\frac{1}{2}n-2} \,. \tag{3.6}$$

At $n \to 4$ this behaves like

$$\tfrac{1}{4}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,\frac{1}{8\pi^2(4-n)} \quad + \text{finite as } n \to 4 \,. \tag{3.7}$$

A counter-term in the Lagrangian $\mathcal{L}[A,x]$ is needed:

$$\Delta\mathcal{L} = \frac{1}{8\pi^2(n-4)}\;\frac{1}{4} M_{ij}[A,x]\, M_{ji}[A,x] \,, \tag{3.8}$$

so that $\mathcal{L} + \Delta\mathcal{L}$ yields a finite amplitude as $n \to 4$. It will be clear that diagrams containing more M vertices give rise to integrals that are finite at $n \to 4$, and no counter-terms with more than two factors M are needed. On the other hand, tadpole diagrams with only one M (or N) vertex are zero because the integrals

$$\int \frac{\mathrm{d}^n k}{k^2 - i\epsilon} \,, \quad \int \frac{\mathrm{d}^n k\, k_\mu}{k^2 - i\epsilon} \,, \tag{3.9}$$

are exactly zero in the n dimensional regularization procedure [9]. Note that eq. (3.8) describes already the complete counter-term in all scalar field theories ($N^\mu_{ij} = 0$). For instance, in ϕ^4

$$\Delta \mathcal{L}[A, x] = \frac{1}{32\,\pi^2 (n-4)} \, (m^2 + \tfrac{1}{2}\lambda A^2(x))^2 \,, \tag{3.10}$$

(where A is the scalar field). Note that in scalar field theories $\Delta\mathcal{L}$ has always the same sign.

4. Symmetry requirements. The pole-algebra for scalar and vector fields

In this section we consider the case

$$W^{\mu\nu}_{ij} = -\delta^{\mu\nu}\delta_{ij}, \qquad N^\mu_{ij} \neq 0 \,, \qquad M_{ij} \neq 0 \,. \tag{4.1}$$

The Lagrangian (2.3) then reads

$$\mathcal{L}[\phi] = -\tfrac{1}{2}(\partial_\mu \phi)^2 + \phi N^\mu \partial_\mu \phi + \tfrac{1}{2}\phi M \phi \,, \tag{4.2}$$

in which we suppressed the indices i, j, and the argument x.

In calculating the poles at dimension $n = 4$ one can make use of the fact that all integrals of the one-loop graphs can be split up into integrals of the type

$$\int \mathrm{d}^n k \frac{k_\mu}{k^2}, \quad \int \frac{\mathrm{d}^n k}{k^2}, \quad \int \mathrm{d}^n k \, \frac{k_\mu}{(k^2)^2} \,, \tag{4.3a}$$

$$\int \frac{\mathrm{d}^n k}{(k^2 + \mu^2)^2} \,, \tag{4.3b}$$

and integrals that converge at $n = 4$. [The quantity μ^2 is left in (4.3b) in order to avoid the infra-red divergency].

Now, all integrals (4.3a) are exactly equal to zero in the dimensional procedure, whereas the integral (4.3b) is

$$\int \frac{\mathrm{d}^n k}{(k^2 + \mu^2)^2} \to 2\,\pi^2 i \, \frac{1}{4-n} \tag{4.4}$$

for $n \to 4$, as long as $\mu^2 \neq 0$.

The convergent integrals have no poles at $n \to 4$. So, only the integrals (4.3b) contribute to the residues of the poles, and the coefficient is simply $2\,\pi^2 i$ according to (4.4), which together with the necessary factor $-(2\,\pi)^{-4} i$ leads to the overall factor $1/8\,\pi^2$ in the amplitude. An important feature of the dimensional regularization technique is that the result of the above procedure is independent of the initial choice of integration variable.

We observe that the substitution (4.4) preserves the scale dimension at $n \to 4$, and a simple power counting argument leads to the most general form that the necessary counter-Lagrangian $\Delta \mathcal{L}$ can have:

$$\Delta \mathcal{L} = \frac{1}{\epsilon}\mathrm{Tr}\ \{a_0 M^2 + a_1(\partial_\mu N_\nu)^2 + a_2(\partial_\mu N_\mu) + a_3 M N^2 + a_4\, N_\mu N_\nu \partial_\mu N_\nu + a_5 (N^2)^2 + a_6 (N_\mu N_\nu)^2\}\,, \tag{4.5}$$

where

$$\frac{1}{\epsilon} = \frac{1}{8\,\pi^2(n-4)} \tag{4.6}$$

and Tr means "trace with respect to the indices i, j" (Note that we required N^μ to be an antisymmetric and M a symmetric matrix).

From the preceding section we know already that $a_0 = \frac{1}{4}$, and by considering the other relevant diagrams one can compute the other coefficients. There is, however, a faster and more elegant way to obtain these coefficients, and that is by performing local gauge transformations. The argument goes as follows (we have assured ourselves of its correctness by also explicitly computing all the diagrams).

Let us rewite the Lagrangian (4.2), noting that N^μ is antisymmetric,

$$\mathcal{L}[\phi] = -\tfrac{1}{2}\,(\partial_\mu \phi + N_\mu \phi)^2 + \tfrac{1}{2}\,\phi X \phi\,, \tag{4.7}$$

with

$$X = M - N_\mu N_\mu\,. \tag{4.8}$$

In this notation it will be clear that $\mathcal{L}(\phi)$ is invariant under

$$\phi'(x) = \phi(x) + \Lambda(x)\,\phi(x)\,, \tag{4.9}$$

$$\left.\begin{aligned} N'_\mu &= N_\mu - \partial_\mu \Lambda + \Lambda N_\mu - N_\mu \Lambda\,, \\ X' &= X + \Lambda X - X\Lambda\,, \end{aligned}\right\} \tag{4.10}$$

where $\Lambda(x)$ is an arbitrary, infinitesimal, antisymmetric matrix. Therefore $\Delta \mathcal{L}$ also will be invariant under (4.10).

The only expression invariant under (4.10) and of dimension four, is

$$\Delta \mathcal{L} = \frac{1}{\epsilon}\ \mathrm{Tr}\ \{a X^2 + b\, Y_{\mu\nu} Y_{\mu\nu}\}\,, \tag{4.11}$$

with

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + N_\mu N_\nu - N_\nu N_\mu\,. \tag{4.12}$$

It follows that (4.5) can be rewritten in this short way; now there are only two independent coefficients, and it will be clear that

$$a = a_0 = \tfrac{1}{4}\,. \tag{4.13}$$

An easy way to calculate the coefficient b is to compute the merely logarithmically divergent diagram with four N legs, thus finding a_5 and a_6:

$$a_5 = \tfrac{1}{6} \quad = a - 2b\,,$$
$$a_6 = \tfrac{1}{12} \quad = 2b\,. \tag{4.14}$$

From this we derive that the complete counter Lagrangian is

$$\Delta \mathcal{L} = \frac{1}{\epsilon}\,\mathrm{Tr}\,\{\tfrac{1}{4} X^2 + \tfrac{1}{24} Y_{\mu\nu} Y_{\mu\nu}\}\,, \tag{4.15}$$

with X and Y as defined in (4.8) and (4.12). Note that the sign is now no longer fixed: $Y_{\mu\nu}$ is antisymmetric in its indices i, j, hence $Y_{\mu\nu}Y_{\mu\nu}$ is always negative.

Formula (4.15) will be used as a building block for further extensions of our algorithm, to include complex fields (sect. 5) and fermion fields (sect. 6).

5. Complex fields

Until now we took all scalar fields to be real. Of course, charged fields can also be split into real components. But it is useful to have the formula for complex fields also (e.g., Faddeev-Popov ghosts in gauge field theories)†.

So, suppose we have the Lagrangian

$$\mathcal{L} = -\partial_\mu \phi^* \partial_\mu \phi + 2\phi^* \mathcal{N}_\mu \partial_\mu \phi + \phi^* \mathcal{M} \phi\,. \tag{5.1}$$

In general we cannot pose further conditions on $\mathcal{N}_\mu$ and $\mathcal{M}$, so

$$\mathcal{N}_\mu \neq \widetilde{\mathcal{N}}_\mu\,; \qquad \mathcal{M} \neq \widetilde{\mathcal{M}} \tag{5.2.}$$

Writing

$$\sqrt{2}\,\phi = \phi_1 + i\phi_2\,,$$
$$\phi_i = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{5.3}$$

one can find the objects N^μ and M of sect. 2 [note that we require the symmetry properties (2.4)]:

$$N^\mu = \tfrac{1}{2}(\mathcal{N}_\mu - \widetilde{\mathcal{N}}_\mu) - \tfrac{1}{2}\sigma_y(\mathcal{N}_\mu + \widetilde{\mathcal{N}}_\mu)\,,$$
$$M = \tfrac{1}{2}(\mathcal{M} + \widetilde{\mathcal{M}}) - \tfrac{1}{2}\partial_\mu(\mathcal{N}_\mu + \widetilde{\mathcal{N}}_\mu)$$
$$\quad - \tfrac{1}{2}\sigma_y(\mathcal{M} - \widetilde{\mathcal{M}}) + \tfrac{1}{2}\sigma_y\,\partial_\mu(\mathcal{N}_\mu - \widetilde{\mathcal{N}}_\mu)\,, \tag{5.4}$$

† Note that in diagrams with charged physical particles, external lines can carry charge also, in which case formula (5.1) is not applicable.

with

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

It is then convenient to define

$$\begin{aligned} \mathcal{X} &= \mathcal{M} - \partial_\mu \mathcal{N}_\mu - \mathcal{N}_\mu \mathcal{N}_\mu , \\ \mathcal{Y}_{\mu\nu} &= \partial_\mu \mathcal{N}_\nu - \partial_\nu \mathcal{N}_\mu + \mathcal{N}_\mu \mathcal{N}_\nu - \mathcal{N}_\nu \mathcal{N}_\mu , \end{aligned} \tag{5.5}$$

because then

$$\begin{aligned} X &= \tfrac{1}{2} (\mathcal{X} + \widetilde{\mathcal{X}}) - \tfrac{1}{2} \sigma_y (\mathcal{X} - \widetilde{\mathcal{X}}) . \\ Y_{\mu\nu} &= \tfrac{1}{2} (\mathcal{Y}_{\mu\nu} - \widetilde{\mathcal{Y}}_{\mu\nu}) - \tfrac{1}{2} \sigma_y (\mathcal{Y}_{\mu\nu} + \widetilde{\mathcal{Y}}_{\mu\nu}) , \end{aligned} \tag{5.6}$$

and $\Delta \mathcal{L}$, formula (4.15) then becomes

$$\Delta \mathcal{L} = \frac{1}{\epsilon} \operatorname{Tr} \{ \tfrac{1}{2} \mathcal{X}^2 + \tfrac{1}{12} \mathcal{Y}_{\mu\nu} \mathcal{Y}_{\mu\nu} \} . \tag{5.7}$$

The factor two difference with eq. (4.15) arises from the doubling of all field components. This equation (5.7) can also very easily be derived from diagrams and symmetry requirements. And then, starting from (5.7) one can re-obtain eq. (4.15). This can be seen by replacing (5.1) by

$$\mathcal{L} = -\partial_\mu \phi^* \partial_\mu \phi + \phi^* (\partial_\mu N_\mu + N_\mu \partial_\mu) \phi + \phi^* M \phi , \tag{5.8}$$

so that

$$\mathcal{M} = M + \partial_\mu N_\mu ; \qquad \mathcal{N}_\mu = N_\mu . \tag{5.9}$$

If

$$N_\mu = -\widetilde{N}_\mu , \qquad M = \widetilde{M} ,$$

then the vertices have the symmetric form like in the case of real fields, and formula (4.15) for real fields follows immediately, after we divided by 2, in order to account for the doubling of the components.

6. Extension to fermions

Our algorithm can also be applied to fermions. Let η be the background fermion field. We write

$$\mathcal{L}[A_i+\phi_i,\bar\eta_j+\bar\psi_j,\eta_k+\psi_k,x] = \mathcal{L}[A,\bar\eta,\eta,x] + \text{terms linear in } \phi, \bar\psi \text{ and } \psi$$

$$-\tfrac{1}{2}(\partial_\mu\phi)^2 + \phi_i N^\mu_{ij}\partial_\mu\phi_j + \tfrac{1}{2}\phi_i M_{ij}\phi_j - \bar\psi\gamma_\mu\partial_\mu\psi + \bar\psi F\psi + \bar\psi\alpha_i\phi_i + \bar\beta_i\psi\phi_i$$

$$+ \text{higher orders in } \phi, \bar\psi \text{ and } \psi. \tag{6.1}$$

Here, again, N, M, F, α and $\bar\beta$ are quantities that contain the background fields A, $\bar\eta$ and η and are therefore space-time dependent. α and $\bar\beta$ are linear combinations of the Fermi fields η and $\bar\eta$ resp. The object F will in general contain γ matrices.

Let us write the fermion propagator as

$$-\frac{i\gamma p}{p^2-i\epsilon}\,. \tag{6.2}$$

Now, if we arrange the factors $-i\gamma p$ into the vertices we can apply the formulae for the bosons. The procedure is made transparent if we substitute into (6.1)

$$\bar\psi\to\bar\psi\,,\qquad \psi\to-\gamma_\mu\partial_\mu\xi\,, \tag{6.3}$$

and, as in sect. 5, we replace the ϕ line by a complex line, bearing in mind that this doubles complete ϕ loops: in other words, we first calculate $\Delta\mathcal{L}$ for

$$\mathcal{L} = -(\partial_\mu+N_\mu)\phi^*(\partial_\mu+N_\mu)\phi + \phi^* X\phi$$

$$-\partial_\mu\bar\psi\partial_\mu\xi - \bar\psi F\gamma_\mu\partial_\mu\xi + \bar\psi\alpha\phi - \bar\beta\phi^*\gamma_\mu\partial_\mu\xi\,, \tag{6.4}$$

where $X = M - N_\mu N_\mu$, as usual.

Taking these fields together in vectors $(\phi^*, \bar\psi)$ and † $(\phi,\xi)^{\mathrm{T}}$, we can define the objects $\mathcal{M}$, $\mathcal{N}_\mu$, $\mathcal{X}$ and $\mathcal{Y}_{\mu\nu}$ of sect. 5:

$$\mathcal{M} = \begin{bmatrix} X+\partial_\mu N_\mu + N_\mu^2 & 0 \\ \alpha & 0\end{bmatrix},$$

$$\mathcal{N}_\mu = \begin{bmatrix} N_\mu & -\tfrac{1}{2}\bar\beta\gamma_\mu \\ 0 & -\tfrac{1}{2}F\gamma_\mu\end{bmatrix},$$

$$\mathcal{X} = \begin{bmatrix} X & \tfrac{1}{2}(\partial_\mu\bar\beta + N_\mu\bar\beta - \tfrac{1}{2}\bar\beta\gamma_\mu F)\gamma_\mu \\ \alpha & \tfrac{1}{2}H\end{bmatrix},$$

$$\mathcal{Y}_{\mu\nu} = \begin{bmatrix} Y_{\mu\nu} & * \\ 0 & -\tfrac{1}{2}H_{\mu\nu}\end{bmatrix}, \tag{6.5}$$

† Superscript T implies transpose.

where

$$H = \partial_\mu F \gamma_\mu - \tfrac{1}{2} F \gamma_\mu F \gamma_\mu \,,$$

$$H_{\mu\nu} = \partial_\mu F \gamma_\nu - \partial_\nu F \gamma_\mu - \tfrac{1}{2} F \gamma_\mu F \gamma_\nu + \tfrac{1}{2} F \gamma_\nu F \gamma_\mu \,, \tag{6.6}$$

and an entry * indicates that it is irrelevant. According to formula (5.7) we find that this Lagrangian (6.4) gives rise to

$$\Delta \mathcal{L} = \frac{1}{\epsilon} \mathrm{Tr} \, \{ \tfrac{1}{2} X^2 + \tfrac{1}{2} (\partial_\mu \bar{\beta} + N_\mu \bar{\beta} - \tfrac{1}{2} \bar{\beta} \gamma_\mu F) \gamma_\mu \alpha$$

$$+ \tfrac{1}{8} H^2 + \tfrac{1}{12} Y_{\mu\nu} Y_{\mu\nu} + \tfrac{1}{48} H_{\mu\nu} H_{\mu\nu} \} \,, \tag{6.7}$$

but now we have to remember the factor $\frac{1}{2}$ to replace the ϕ fields by real fields, and a minus sign for the Fermion loop. Further, we rewrite the $\bar{\beta}\alpha$ term in a more familiar way. Our result for the original Lagrangian (6.1) reads

$$\Delta \mathcal{L} = \frac{1}{\epsilon} \mathrm{Tr} \, \{ \tfrac{1}{4} X^2 + \tfrac{1}{24} Y_{\mu\nu} Y_{\mu\nu} - \tfrac{1}{2} \bar{\beta} \gamma_\mu (\partial_\mu \alpha + \tfrac{1}{2} F \gamma_\mu \alpha + N_\mu \alpha)$$

$$- \tfrac{1}{8} H^2 - \tfrac{1}{48} H_{\mu\nu} H_{\mu\nu} \} \,. \tag{6.8}$$

The trace of the last two terms in (6.8) is one over spinor indices. The occurrence of numerous gamma matrices in (6.6) and (6.8) makes calculations with these formulae rather lengthy. A nice feature of our algorithm however, is that one can calculate the traces of these series of gamma matrices once and forever.

As long as no exotic gauge is chosen, one can write F in any renormalizable theory as

$$F = S + P \gamma_5 + V_\mu \gamma_\mu + A_\mu \gamma_\mu \gamma_5 \,, \tag{6.9}$$

where the new quantities may still be matrices, but do not contain spinor indices. V_μ and A_μ are gauge fields, multiplied with matrices according to the group representations.

It is natural to define now the covariant derivatives:

$$D_\mu S = \partial_\mu S - V_\mu S + S V_\mu + A_\mu P + P A_\mu \,,$$

$$D_\mu P = \partial_\mu P - V_\mu P + P V_\mu + A_\mu S + S A_\mu \,,$$

$$D_\mu \alpha = \partial_\mu \alpha + (-V_\mu + A_\mu \gamma_5) \alpha + N_\mu \alpha \,, \tag{6.10}$$

and right- and left-handed gauge fields:

$$A^{\mathrm{L}}_\mu = A_\mu - V_\mu, \quad A^{\mathrm{R}}_\mu = A_\mu + V_\mu,$$

$$G^{\mathrm{L}}_{\mu\nu} = \partial_\mu A^{\mathrm{L}}_\nu - \partial_\nu A^{\mathrm{L}}_\mu - A^{\mathrm{L}}_\mu A^{\mathrm{L}}_\nu + A^{\mathrm{L}}_\nu A^{\mathrm{L}}_\mu \,, \tag{6.11}$$

same for $G^{\mathrm{R}}_{\mu\nu}$.

It is now possible to rewrite formula (6.8), calculating all occurring traces over arrays of gamma matrices:

$$\Delta \mathcal{L} = \frac{1}{8\pi^2(n-4)} \quad \mathrm{Tr}\, \{ \tfrac{1}{4} X^2 + \tfrac{1}{24} Y_{\mu\nu} Y_{\mu\nu} - \tfrac{1}{2} \bar{\beta} \gamma_\mu D_\mu \alpha$$

$$- \bar{\beta}(S - P\gamma_5)\alpha - (D_\mu S)^2 + (D_\mu P)^2 - (S^2 - P^2)^2$$

$$- (SP - PS)^2 + \tfrac{1}{6} G^{\mathrm{R}}_{\mu\nu} G^{\mathrm{R}}_{\mu\nu} + \tfrac{1}{6} G^{\mathrm{L}}_{\mu\nu} G^{\mathrm{L}}_{\mu\nu} \}\,, \tag{6.12}$$

with the definitions (6.1), (4.8), (4.12), (6.9), (6.10) and (6.11).

It is invariant under local chiral gauge transformations †.

7. Use of the equation of motion

In the foregoing we derived the prescription to calculate $\Delta \mathcal{L}\,(n)$, such that

$$\mathcal{L} + \Delta \mathcal{L}\,(n)$$

leads to finite Green functions in the limit $n \to 4$, up to the order of one-loop graphs. $\Delta \mathcal{L}$ is proportional to some coupling constant λ in the theory, and we persistently disregarded corrections of order λ^2 since these are determined also by the much more complicated two-loop diagrams.

Suppose we make a renormalization of the fields A_i of the type

$$A_i(x) \to A_i(x) + \Delta A_i(x)\,, \tag{7.1}$$

where $\Delta A_i(x)$ is some function of the fields $A(x)$ and is also proportional to λ.

In terms of the new fields the Lagrangian becomes

$$\mathcal{L} \to \mathcal{L} + \mathcal{L}'_i\,[A,x]\;\Delta A_i(x) + \mathrm{O}(\lambda^2)\,. \tag{7.2}$$

If $\Delta A_i(x)$ in (7.1) contains poles at $n \to 4$ then the Green functions will become infinite, in general.

The S-matrix elements, however, if defined from these Green functions with the proper external line renormalization factors, will not be influenced by the replacement (7.1), because the external line renormalization factors will cancel the effects from (7.1).

From this it follows that any change in $\Delta \mathcal{L}$ which is proportional to $\mathcal{L}'[A,x]$ does not influence the S-matrix. In other words: one may make use of the equations

† Chiral anomalies do not occur in these calculations since these anomalies are finite as $n \to 4$. We only consider residues of the poles at $n \to 4$. Our definition of γ_5 is, also at $n \neq 4$: $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. At first sight one might also expect a term like $\mathrm{Tr}\,\epsilon_{\mu\nu\alpha\beta}(G^{\mathrm{L}}_{\mu\nu}G^{\mathrm{L}}_{\alpha\beta} - G^{\mathrm{R}}_{\mu\nu}G^{\mathrm{R}}_{\alpha\beta})$, but this quantity is a total derivative.

of motion

$$\mathcal{L}_i' [A, x] = 0 \,, \tag{7.3}$$

to simplify our expressions for $\Delta \mathcal{L}$. The Green functions will then no longer remain finite, but the S-matrix, if properly defined, will remain finite.

In the covariant background field renormalization method for gauge theories, full advantage is made of eq. (7.3) for the background fields [6, 7]. We show this in an example in sect. 8.

8. Examples

8.1. ϕ^4 theory with N components

Let us take

$$\mathcal{L} = \sum_{i=1}^{N} \left[-\tfrac{1}{2} (\partial_\mu \phi_i)^2 - \tfrac{1}{2} m^2 \phi_i^2\right] - \tfrac{1}{8} \lambda (\Sigma_i \phi_i^2)^2 \,. \tag{8.1}$$

From the definition (2.1) and (4.8) we find

$$M_{ij} = X_{ij} = -(m^2 + \tfrac{1}{2} \lambda \sum_k \phi_k^2) \delta_{ij} - \lambda \phi_i \phi_j \,, \tag{8.2}$$

$$N_{ij}^\mu = 0 \,.$$

From formula (4.15) we find

$$\Delta \mathcal{L} = \frac{1}{\epsilon} \operatorname{Tr} \tfrac{1}{4} X^2$$

$$= \frac{1}{\epsilon} [\lambda(1 + \tfrac{1}{2} N) \tfrac{1}{2} m^2 \phi^2 + (4 + \tfrac{1}{2} N) \tfrac{1}{8} \lambda^2 (\phi^2)^2 + \text{const}] \,, \tag{8.3}$$

where

$$\frac{1}{\epsilon} = \frac{1}{8 \pi^2 (n - 4)} \,. \tag{8.4}$$

Evidently, there is a first order mass renormalization

$$\Delta m = -\frac{1}{2\epsilon} \lambda (1 + \tfrac{1}{2} N) m \,, \tag{8.5}$$

and a coupling constant renormalization

$$\Delta \lambda = -\frac{1}{\epsilon} \lambda^2 (4 + \tfrac{1}{2} N) \,, \tag{8.6}$$

and there is no first order field renormalization. Note that we found here the first coefficients of the expansions discussed in ref. [4]:

$$m_{\mathrm{B}} = m + \Delta m\,,$$

$$\lambda_{\mathrm{B}} = \lambda + \Delta\lambda\,. \tag{8.7}$$

8.2. Pure Yang-Mills fields, of a semi-simple group

In this example we demonstrate the elegance of the background field quantization method [6, 7]. We start with the gauge invariant Lagrangian

$$\mathcal{L}^{\mathrm{inv}}[W] = -\tfrac{1}{4}\, G^a_{\mu\nu}\, G^a_{\mu\nu}\,. \tag{8.8}$$

with

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu\,,$$

where f_{abc} is completely antisymmetric. As in refs. [6, 7], we postpone the subsidiary gauge condition. First we write

$$W_\mu \to W_\mu + \phi_\mu\,, \tag{8.9}$$

and, as always, expand $\mathcal{L}$ up to the second order in ϕ:

$$G^a_{\mu\nu} \to G^a_{\mu\nu} + D_\mu \phi^a_\nu - D_\nu \phi^a_\mu + g f_{abc}\, \phi^b_\mu \phi^c_\nu\,, \tag{8.10}$$

where

$$D^{ac}_\mu = \partial^{ac}_\mu + g f^{abc}\, W^b_\mu \tag{8.11}$$

is the covariant derivative (note that D_μ contains the background field only). From sect. 7 we know that we may assume that the background field satisfies the equation of motion, hence

$$\mathcal{L}^{\mathrm{inv}} \to \mathcal{L}^{\mathrm{inv}}[W] - \tfrac{1}{2}\,(D_\mu \phi_\nu)^2 + \tfrac{1}{2}\,(D_\mu \phi_\nu)^2$$

$$- g\, G^c_{\alpha\beta} f_{abc}\, \phi^a_\alpha \phi^b_\beta + \text{higher orders in } \phi\,. \tag{8.12}$$

Here we made use of

$$(D_\mu D_\nu)^{ac} - (D_\nu D_\mu)^{ac} = g f_{abc}\, G^b_{\mu\nu}\,. \tag{8.13}$$

The Lagrangian (8.12) is invariant under the infinitesimal gauge transformation:

$$\phi^a_\mu \to \phi^a_\mu + g f^{abc}\, \Lambda^b (\phi^c_\mu + W^c_\mu) - \partial_\mu \Lambda^a$$

$$\to \phi^a_\mu + g f^{abc}\, \Lambda^b \phi^c_\mu - D_\mu \Lambda^a\,. \tag{8.14}$$

The quantization procedure for ordinary gauge field theories can be applied [3]. The gauge term

$$-\tfrac{1}{2}\,(\partial_\mu W_\mu + \partial_\mu \phi_\mu)^2 \tag{8.15}$$

would lead to the usual Feynman rules in the Feynman gauge. But it is more convenient to choose [6]

$$-\tfrac{1}{2}\,(D_\mu \phi_\mu)^2 \,. \tag{8.16}$$

The relevant Lagrangian for the vector field is then

$$\mathcal{L} = -\tfrac{1}{2}\,(D_\mu \phi_\nu)^2 - g\, G^c_{\alpha\beta} f_{abc}\, \phi^a_\alpha\, \phi^b_\beta \,, \tag{8.17}$$

and comparing this with formula (4.7) we find

$$\begin{aligned} N^{\mu\,ab}_{\;\;\alpha\beta} &= g f^{acb}\, W^c_\mu\, \delta_{\alpha\beta} \,, \\ X^{ab}_{\alpha\beta} &= -\,2g f^{abc}\, G^c_{\alpha\beta} \,, \\ Y_{\mu\nu\,\alpha\beta}^{\;\;ab} &= g f^{acb}\, G^c_{\mu\nu}\, \delta_{\alpha\beta} \,. \end{aligned} \tag{8.18}$$

Formula (4.15) gives us directly the contribution of the vector fields

$$\Delta\mathcal{L}^{\text{vector}} = \frac{1}{\epsilon}\, C\, \tfrac{5}{6}\, g^2\, G^a_{\mu\nu}\, G^a_{\mu\nu} \,, \tag{8.19}$$

where C is defined by

$$f_{abc}\, f_{abd} = C\, \delta_{cd} \,. \tag{8.20}$$

In this gauge however, there is also a contribution from the Faddeev-Popov ghost [3]. From the gauge transformation law (8.14) we find the ghost Lagrangian

$$\mathcal{L}^{\text{Fadd.-Popov}} = -D_\mu\, \phi^*_a\, D_\mu\, \phi_a + \text{irrelevant interactions with } \phi^a_\mu \,. \tag{8.21}$$

The fields ϕ_a in (8.21) are complex scalars with isospin 1. To calculate its contribution we use the results of sect. 5

$$\begin{aligned} \mathcal{X}^{ab} &= 0 \,, \\ \mathcal{Y}^{ab}_{\mu\nu} &= g f^{acb}\, G^c_{\mu\nu} \,. \end{aligned} \tag{8.22}$$

Taking its Fermi statistics into account, we find the contribution

$$\Delta\mathcal{L}^{\text{ghost}} = +\frac{1}{\epsilon}\, C\, \tfrac{1}{12}\, g^2\, G^a_{\mu\nu}\, G^a_{\mu\nu} \,. \tag{8.23}$$

Together with (8.19) we find the complete counterterm

$$\Delta\mathcal{L} = \frac{1}{\epsilon}\, C\, \tfrac{11}{12}\, g^2\, G^a_{\mu\nu}\, G^a_{\mu\nu} \,. \tag{8.24}$$

Writing

$$\mathcal{L}[W,g] + \Delta\mathcal{L} = \mathcal{L}[W + \Delta W, g + \Delta g] \,, \tag{8.25}$$

we find the first order necessary field renormalization

$$\Delta W^a_\mu = -\frac{1}{\epsilon} C g^2 \tfrac{11}{24} W^a_\mu \,, \tag{8.26}$$

and the coupling constant renormalization

$$\Delta g = \frac{1}{\epsilon} C \tfrac{11}{24} g^3 \,. \tag{8.27}$$

Note the difference in sign in comparison with eq. (8.6). The field renormalization (8.26) is different in other gauges, but (8.27) is gauge independent. The sign in (8.27) is such that the large momentum limit of Green's functions is calculable, but there is an infra-red catastrophe [4, 5].

The fermion contribution can readily be obtained from eq. (6.12). Note that in eq. (6.12) $G^{\mathrm{R,L}}_{\mu\nu}$ still contain the matrices of the group representation and are therefore antihermitean. Hence the sign of the fermion contribution will always be opposite to the gauge field contribution (8.24). This phenomenon has been noted before [5]. It is also the reason why the sign of the coupling constant renormalization in QED is opposite to that of pure Yang-Mills fields [eq. (8.27)], because in QED only fermions contribute.

By giving the fermions in a gauge theory the appropriate multiplicity one can even obtain theories with $\Delta\mathcal{L} = 0$, or one can choose the one-loop contribution to $\Delta\mathcal{L}$ very small, such that it cancels all higher order contributions. This way a calculable conformally invariant theory can be obtained.

9. Quantum gravity

Our algorithm can be applied to study the infinity structure of the quantum theory of gravity. For that end, however, it is still incomplete: we should study the case

$$W^{\mu\nu}_{ij} = g^{\mu\nu}\,\delta_{ij} \neq \delta^{\mu\nu}\,\delta_{ij} \tag{9.1}$$

in formula (2.1). For the rest, the calculation of the one-loop infinities goes along the same lines as in the example of subsect. 8.2, where we studied Yang-Mills fields. We postpone the details of our calculations to a separate publication [8], but we shall mention the result here.

Using the invariant quantization procedure, we get an invariant result for the pole parts of the one-loop graphs. In a theory with gravity alone, one derives easily from power counting, that the only candidates for $\Delta \mathcal{L}$ are [9]

$$\sqrt{g} R_{\mu\nu} R_{\mu\nu}, \qquad \sqrt{g} R^2 , \tag{9.2}$$

where $g = \det (g_{\mu\nu})$ and $R_{\mu\nu}$ and R are the contracted Riemann tensors. However, from sect. 7 we know that the equation of motion may be substituted, which is

$$R_{\mu\nu} = 0 , \qquad R = 0 , \tag{9.3}$$

and we derive immediately that there is no physical infinity in the one-loop quantum corrections to graviton-graviton scattering!

Stated differently, the counter terms (9.2) can be absorbed by a "renormalization" of the matric tensor $g_{\mu\nu}$:

$$g_{\mu\nu} \to g_{\mu\nu} + \frac{1}{n-4} (\alpha R_{\mu\nu} + \beta R g_{\mu\nu}) , \tag{9.4}$$

where α and β are unobservable coefficients.

The situation becomes different if we add to the theory other fields, which carry energy momentum, such that eq. (9.3) is no longer true. We studied the simplest example of such a case: one massless Klein-Gordon field ϕ interacting with gravity. We started with the Lagrangian

$$\mathcal{L} = -\sqrt{g} R - \tfrac{1}{2} \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi . \tag{9.5}$$

After insertion of the equation of motion only one gauge invariant term survives in $\Delta \mathcal{L}$. By means of our algorithm we calculated its coefficient and found

$$\Delta \mathcal{L} = \frac{1}{\epsilon} \frac{203}{80} \sqrt{g} R^2 . \tag{9.6}$$

It is impossible to renormalize this term away by renormalization of fields or physical parameters, so it survives as a real infinity. We conclude that the theory of bare gravity is one-loop renormalizable, but if matter is added in the form of Klein-Gordon fields, physical divergencies remain. If other fields or other interactions are added in $\mathcal{L}$, the number of possible terms in $\Delta \mathcal{L}$ increases rapidly and only miraculous cancellations could restore renormalizability. In particular, use of the "improved energy momentum tensor" of ref. [10] does not improve the situation here.

We wish to thank Professor M. Veltman, who carefully checked all calculations in this paper.

References

[1] G. 't Hooft and M. Veltman, Nucl. Phys. B44 (1972) 189.
[2] C.G. Bollini and J.J. Giambiagi, Phys. Letters 40B (1972) 566;
J.F. Ashmore, Nuovo Cimento Letters 4 (1972) 289.
[3] G. 't Hooft and M. Veltman, Nucl. Phys. B50 (1972) 318;
G. 't Hooft and M. Veltman, Proc. Marseille Conf., 19–23 June 1972, CERN preprint TH. 1571.
[4] G. 't Hooft, CERN preprint TH. 1666, to be published in Nucl. Phys.
[5] G. 't Hooft, Marseille Conf., 19–23 June 1972, unpublished;
H.D. Politzer, Harvard preprint (1973);
D.J. Gross and F. Wilczek, Princeton preprint (1973);
S. Coleman and D. Gross, Princeton preprint (1973).
[6] J. Honerkamp, Nucl. Phys. B48 (1972) 269;
J. Honerkamp, Proc. Marseille Conf., 19–23 June 1972, CERN preprint TH. 1558.
[7] B.S. DeWitt, Phys. Rev. 162 (1967) 1195, 1239, and Dynamical theory of groups and fields, in relativity, groups and topology, Les Houches (Gordon and Breach, New York, 1973).
[8] G. 't Hooft and M. Veltman, CERN preprint TH. 1723.
[9] D.M. Capper and G. Leibbrandt, ICTP Trieste preprint IC/72/158.
[10] C.G. Callan, S. Coleman and R. Jackiw, Ann. of Phys. 59 (1970) 42.