The evolution of gravitating point particles in 2+1 dimensions

1993 Class. Quantum Grav. 10 1023

(http://iopscience.iop.org/0264-9381/10/5/019)

Class. Quantum Grav. 10 (1993) 1023–1038. Printed in the UK

The evolution of gravitating point particles in 2+1 dimensions

G 't Hooft
Institute for Theoretical Physics, University of Utrecht, PO Box 80006, 3508 TA Utrecht, The Netherlands

Received 26 January 1993

Abstract. The complete history of a set of classical gravitating point particles in 2+1 dimensions is considered, in the absence of a cosmological constant. We formulate the equations of motion in terms of a time-dependent tessellation of Cauchy surfaces, of which a number of examples were run on a computer. In particular we focus on the initial and final states. A given universe may either continue to expand for ever or shrink to a point in a final crunch, the latter being the rule rather than an exception. The past history may either be a 'big bang' or an infinitely large shrinking universe. Universes with $g = 0$ may have both a bang in the past and a crunch in the future. Universes with $g \leqslant 1$ and neither a bang in the past nor a crunch in the future were not found. Our findings must have important consequences for any possible quantized version of such a system.

PACS number: 0420

1. Introduction

Einstein gravity in $2+1$ dimensions is nearly trivial. Spacetime surrounding a point particle is locally flat, so that there are no tidal forces and no gravitons; particles move in a straight line, and hence the theory is 'exactly solvable' [1].

Nevertheless, there are surprises. The first signs of unexpected behaviour were found by Gott [2], who indicated that an isolated pair of point particles, moving rapidly with respect to each other, may generate a surrounding region where closed timelike curves occur. Since this is physically unacceptable (at least to this author), a mechanism was searched for that allows nature to avoid such things from happening spontaneously. This mechanism was soon found. The reason why regions of spacetime with closed timelike curves cannot be generated in any universe with physically acceptable boundary conditions is that a Gott pair of particles leads to a 'big crunch' [3]: spacetime shrinks to a point 'before' any closed timelike curve has a chance of opening up. Any open universe with a Gott pair in it must have physically unacceptable boundary conditions (a closed timelike loop at the boundary). We here consider only closed spaces.

But the precise way in which a big crunch arises, in particular the question of how particles that start off moving in essentially arbitrary directions nevertheless manage to approach each other arbitrarily closely in a finite timespan, still remained something of a mystery. Roughly, it seemed as if the particles in this universe behave like a ping-pong ball bouncing back and forth between the table and a bat pushed towards the table. The author ran a few examples of model universes on a computer and was startled at what came out: even a five-particle universe could exhibit a tremendous amount of complexity. Hundreds of cross-overs may take place before a final configuration is settled for, and then this final configuration, the ultimate path towards the big crunch, is also highly interesting. In spite of the absence of a direct Newtonian attraction in $2+1$ dimensions, particles not only exchange energy as soon as they move, but this energy rapidly increases when the big crunch is set in.

Indeed, as the crunch approaches all particles undergo a tremendous Lorentz boost. Now pairs of particles not only meet each other infinitely many times with ever decreasing impact parameters, but also with a centre-of-mass energy that increases exponentially with a power of $1/(t-t_0)$. This is much more violent than a bouncing ping-pong ball! the reason why this is so is that one should consider the bat moving towards the table with a velocity that keeps pace with that of the ping-pong ball itself.

A description of the tessellation of 2-space and its evolution in time was given in [3]. Several minor details have to be added to that before one can run the system on a computer. Therefore we review this procedure in section 2, and list all possible evolution events in section 3. What has to be added to the original prescription of [3] is that a polygon may shrink to a point and disappear, and furthermore that there may be certain kinds of 'non-transitions' if parts of a universe are expanding and a part is shrinking. None of these refinements have any effects on the conclusions or any other arguments in [3], but could cause havoc if forgotten to be taken into account in a computer program. Another danger in a computer program is that it turns out to be possible to generate 'self-overlapping polygons', i.e. polygons that occupy more than one Riemann sheet of a flat surface. But these are rare, and do not affect any of our arguments.

We describe the initial states and the global spacelike topology for some universes in section 4. Then we briefly discuss what happens if one simulates the evolution towards the future and the past on a computer, in section 5. The asymptotic state, in particular the crunch, is described in section 6, where we prove its asymptotic nature. The system becomes obviously 'ergodic' in the sense that the configurations become infinitely sensitive to the initial parameters.

The examples we studied were spacetimes with topology $\Sigma \times R(1)$, where $\Sigma = S_2$ (so that $g=0$) or $\Sigma = S_1 \times S_1$ (or $g=1$). We then found bounds in the $R(1)$ space, either in the past, or in the future, or both. We could not find a $g=0$ universe with a completely unlimited time interval $R(1)$.

One may suspect that our findings should be extremely important for understanding how the system should be quantized. Attempts to formulate a quantum version of this theory should shed interesting light on many conceptual difficulties in quantum cosmology in general. We discuss our conclusion in section 7.

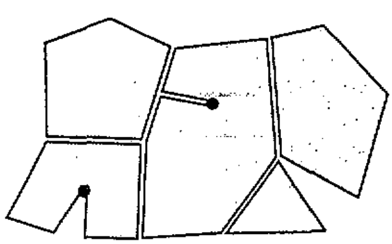

Figure 1. Part of a Cauchy surface. The darker dots are point particles. In this picture one of them has a negligible mass.

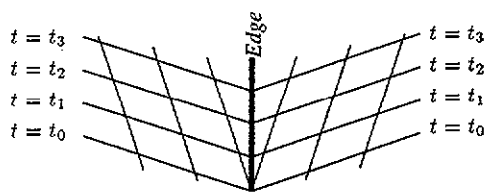

Figure 2. The boundary between two polygons. The preferred Lorentz frames of both polygons are sketched. In this picture the edge moves outwards with respect to both frames.

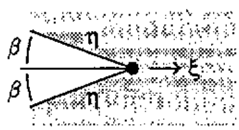

Figure 3. Wedge near a moving particle.

2. The tessellation

Our method of handling universes in 2+1 dimensions such that the causal order of spacetime points remains evident was described in [3]. Here we briefly repeat the basic procedure as formulated in that paper. A Cauchy surface corresponding to a time variable $t = t_1$ is constructed out of locally flat regions of 2-space (polygons). In 3-space the surface $t = t_1$ generates a locally preferred Lorentz frame. The polygons are glued together at boundaries that we call 'seams' or 'edges'—see figure 1. A point particle will in general be situated somewhere inside a polygon, where it produces a wedge.

The transition from one polygon to the next when we cross a seam will be associated with a Lorentz transformation. We shall insist that there should be no time jump across a seam; fortunately we can easily ensure the absence of time jumps by choosing the velocities of an edge, as seen in the frames of the two bordering polygons, to be equal and in opposite directions—see figure 2.

A particle may be moving in time, but then the wedge must always be chosen such that the velocity vector is along the diagonal of the wedge. The wedge is opening up or closing in, so that when a local observer crosses the wedge he also undergoes a Lorentz transformation.

We indicate the velocity of an edge with respect to the fixed Lorentz frame of a polygon as $\tanh\eta$, where η is a Lorentz boost parameter. The relative Lorentz transformation between the two adjacent polygons is then generated by 2η.

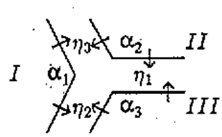

Figure 4. Vertex between three polygons.

Consider first the wedge produced by a moving particle—see figure 3. Let 2β be the opening angle of the wedge. Let $\tanh\xi$ be the velocity of the particle (moving in the direction of the arrow). Let 2η be the Lorentz boost parameter across the wedge. Then the velocity of the edge there is $\tanh\eta$. Suppose that the wedge of a particle at rest has

$$\beta_0 = \pi m\,. \tag{2.1}$$

To describe a moving particle we Lorentz-boost this:

$$\tan\beta = \cosh\xi\ \tan(\pi m)\,. \tag{2.2}$$

Geometry determines the ratio of the velocities:

$$\tanh\eta = \sin\beta\ \tanh\xi\,. \tag{2.3}$$

Eliminating ξ out of this we get the useful relation

$$\cos\beta\ \cosh\eta = \cos(\pi m)\,. \tag{2.4}$$

In general there won't be a particle at a vertex where three polygons meet. Spacetime is locally flat there. Let I, II and III be the three polygons. The Lorentz transformation from I to II is completely determined by giving the orientation of the seam between them and the Lorentz boost parameter 2η. Similarly the transformation from II to III. Clearly we get the Lorentz transformation from I to III by multiplying these. Hence both the orientation angles and the boost parameter for the edge between I and III are determined by the other edges. This gives us relations between the angles α_i and the boost parameters η_i at a vertex. We define these numbers cyclically, as indicated in figure 4.

Defining

$$\sin\alpha_i = s_i \qquad \cos\alpha_i = c_i \qquad \sinh 2\eta_i = \sigma_i \qquad \cosh 2\eta_i = \gamma_i \tag{2.5}$$

we get

$$s_1 : s_2 : s_3 = \sigma_1 : \sigma_2 : \sigma_3 \tag{2.6}$$

$$\gamma_2 s_3 + s_1 c_2 + c_1 s_2 \gamma_3 = 0 \tag{2.7}$$

$$c_1 = c_2 c_3 - \gamma_1 s_2 s_3 \tag{2.8}$$

$$\gamma_1 = \gamma_2\gamma_3 + \sigma_2\sigma_3 c_1 \tag{2.9}$$

$$\cot\alpha_2 = -\cot\alpha_1\ \cosh 2\eta_3 - \coth 2\eta_2\ \sinh 2\eta_3/\sin\alpha_1 \tag{2.10}$$

and all cyclic permutations.

One may recognize these equations as the relations between sides and angles of a triangle in a hyperbolic space, except that the sides η_i may have positive or negative lengths. One derives from (2.8) and (2.6) that the sum of the angles α_i exceeds 360° if all η_i have the same sign, whereas it is less than 360° whenever one sign is different from one other. A remarkable consequence of this is that, by keeping all polygons convex ($s_i > 0$) and all η of the same sign, one can build up Cauchy surfaces with negative curvature, so that surfaces with high Euler characteristics are always possible. Particles contribute positively to the curvature (unless the gravitational constant is negative, a case that we shall not consider), so by adding particles we can build a compact universe with the topology of a 2-sphere. By allowing sign differences in the η_i and non-convex polygons, one also can produce positive curvature locally, but it appears that without particles one cannot build an S_2 space, because clashes arise.

It will be clear that unless all η_i vanish the polygon shapes and sizes will be time dependent. The time derivative $\dot{L}$ of an edge between two vertices A and B gets contributions from both vertices,

$$\dot{L} = g_{\mathrm{A}} + g_{\mathrm{B}} \tag{2.11}$$

where at vertex A in direction 1 the quantity $g_{\mathrm{A},1}$ is given by

$$g_{\mathrm{A},1} = (v_1 \cos\alpha_3 + v_2)/\sin\alpha_3 = (v_1 \cos\alpha_2 + v_3)/\sin\alpha_2 \tag{2.12}$$

where

$$v_i = \tanh\eta_i = \sigma_i/(1+\gamma_i)\,. \tag{2.13}$$

At a particle P the contribution to the time dependence $\dot{L}$ is

$$g_{\mathrm{P}} = \tanh\eta\, \cot\beta\,. \tag{2.14}$$

The topological structure of a tessellation will be denoted by a diagram indicating the edges of the polygons without bothering about actual lengths or angles. Depending on the global topology of 2-space the diagram should be seen as living on a topologically non-trivial sheet, which we unfold by removing a few points. We should then indicate how the sides of the diagram should be identified.

3. Transitions

We now wish to see how such a Cauchy surface may evolve in time. It will be argued that there is a quite natural way to construct the Cauchy surfaces at later times. Strictly speaking the problem is not yet well formulated, because there will be very many different ways to construct such surfaces. But if we ask for the 'simplest' construction the answer will be unique, even though the series obtained will not be time-reversal invariant. The equations for the particles are of course time-reversal invariant, but our description of the solution not, or not quite.

During short intervals of time we may simply allow time to evolve equally fast on all polygons, so that the edges move with their well defined velocities. But it will be unavoidable that as time continues something will happen. It could be that the length of an edge shrinks to zero. It could also happen, since many polygons are not convex, that one of the vertices of a polygon hits one of the other edges, at which point it also becomes illegal to continue the description in terms of these particular polygons. A transition in terms of another set of polygons takes place. It is the succession of many such transitions that we will study.

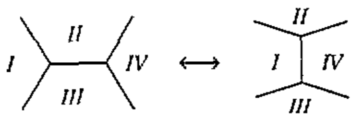

Figure 5. Diagram for transition A.

3.1. Transition type A: exchange

This is a transition that may take place if the length of one of the edges has shrunk to zero. Previously polygons II and III have a boundary in common; after the transition I and IV share a boundary. Since the relative Lorentz transformation between I and IV is already determined before the transition, it must be so that the location and velocity of the new boundary is determined by the geometry before the transition. This is indeed the case. One simply has to apply the triangle equations (2.6)–(2.10) to the two new vertices.

3.2. Transition type B: hop

If the length of the edge of a wedge goes to zero it means that the particle has reached the boundary of a polygon and is ready to hop over to an adjacent polygon, where it will open up a new wedge. Again, the parameters of the new wedge are determined by the triangle relations.

Figure 6. Diagram for transition B.

3.3. Transition type C: split due to particle

Our polygons need not be convex. Any particle with mass $< \frac{1}{2}$ will be associated with a wedge of the shape given in figure 7(a), so that the polygon is indented. Now if the particle reaches the other side it will pass on to an adjacent polygon. The original one will be left split in two (figure 7(b)). Note that, unless the particle hits the edge head-on, not only a new wedge, but also a new boundary AB will emerge. For both of these the triangle relations will determine how they are oriented and how they will move in time.

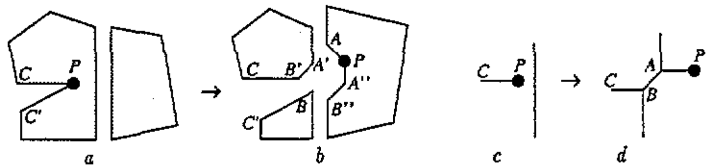

Figure 7. (a), (b) Transition C. The points A, A′ and A″ are to be identified. (c), (d) The same transition in diagrams.

3.4. Transition type D: split due to vacuum vertex

At a vacuum vertex, described by the triangle equations (2.6)–(2.10) one may also encounter angles exceeding 180°. This happens then and only then if the Lorentz boost parameters η at two of the edges have different signs. In this case such a corner may also hit an opposite edge of a polygon, in which case it also splits—see figure 8. Again two new edges appear, whose parameters are determined by the triangle equations.

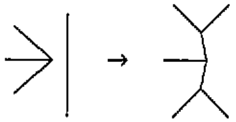

Figure 8. Diagram for transition D.

3.5. Transition type E: particle grazing

In the case of non-convex corners something else may happen. If an edge shrinks to zero it might then be that a transition of the type A or B does not take place. Instead, a particle may 'graze' against a corner—see figure 9. Because one angle exceeds 180° the particle cannot enter the other polygon. We get a sign flip: the sign of the velocities at the wedge and the signs of $\sin\alpha_i$, where α_i are the two adjacent angles, switch in such a way that the triangle relations (2.6)–(2.10) remain valid. Note that (2.9) and (2.10) determine the parameters of a new edge but not yet this sign.

Figure 9. Transition E and its diagrammatic notation.

Usually the sign is determined unambiguously by the demand that no more than one angle at a vertex is allowed to exceed 180°, but otherwise it is fixed by demanding that a new boundary should always start out by growing, never by shrinking to a length less than zero. In the case of transition type E the topology of the diagram does not alter.

3.6. Transition type F: vacuum vertex grazing

The same may happen if a vacuum vertex 'grazes' against a corner. The diagrammatic notation is given in figure 10. Again the topology of the diagram does not change.

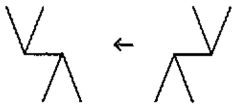

Figure 10. Transition F.

It may be observed that whenever a transition of the type C or D takes place the total number of polygons in our tessellation increases by one. At first sight one might suspect therefore that the total number of polygons will increase indefinitely. Indeed we cannot know in advance how large the number of polygons will become. In general, however, the number stays limited, and this is because of two more possible transistions.

3.7. Transition G: triangle disappearance

In between the transitions the lengths of the edges vary continuously, but the relative angles remain fixed. Hence, if a polygon has the shape of a triangle and if one of its edges then shrinks to zero, the other two edges will shrink to zero simultaneously. This event is simply an end-point of a 3-simplex in spacetime. To continue the tessellation for the future Cauchy surfaces is then very simple: we just remove this triangular polygon—see figure 11.

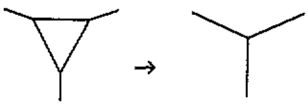

Figure 11. Transition G.

3.8. Transition H: quadrangle disappearance

In general, if a polygon has more than three sides, the angles alone do not determine the ratios of the lengths of the sides unambiguously. If such a polygon shrinks one of the sides will go to zero first. So this polygon may only disappear by first making one of the previously mentioned transitions to become a triangle, and then be subjected to transition G. However, there is an exception. If one of the corners is actually a particle the two adjacent edges form a wedge and hence must be of equal length. Now the ratios of the lengths *are* determined by the angles, and this polygon may shrink conformally to zero. This is transition H.

Note that this quadrangular polygon before disappearance was completely surrounded by only one other polygon. Topologically the transition is given in figure 12. One may now wonder whether there are more shapes of polygons which will shrink conformally to one point. The answer is no, not without an infinitely fine-tuned initial condition. Pentagons with two particles, for instance, cannot occur. Topologically this configuration is impossible, as it is not difficult to verify by attempting to write the corresponding diagram. Also, totally disconnected pieces of a diagram cannot appear.

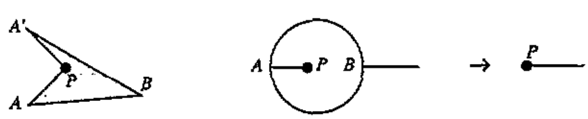

Figure 12. Transition H. Left: a typical quadrangle just before disappearance.

3.9. Transition J: double triangle

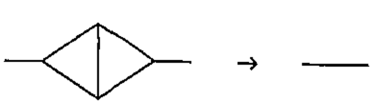

Figure 13. Transition J.

Yet there is still one other possible transition. There may be two adjacent triangles. If one of these disappears the other must disappear at the same instant, leaving just a single edge. It is important to know that such a transition can occur when one wishes to perform a computer simulation—see figure 13.

The above nine topologically distinct kinds of transitions are all needed for following a Cauchy surface through time. We see that the number of polygons will not be constant. It may increase or decrease. Before we can turn this algorithm into a computer program a few more things have to be considered.

4. Degrees of freedom and topology

Given a closed universe with a certain positive number N of point particles in it, we now ask how to construct any one Cauchy surface through a given spacetime point A. Let us choose the local Lorentz frame in A as well, so we know how the surface goes near the point A. We extend the surface in all directions until no more points can be added that are not timelike-separated from any other point already on the surface. We then get a surface with a boundary, but to cross the boundary one would in general need time jumps. By moving the boundaries around one can minimize the time jumps at the boundaries. It is not hard to convince oneself that the end result of this procedure is a surface with no time jumps at all at its boundaries. There is an identification prescription at the boundaries which fixes the overall topological characteristics of this spacelike part of the universe. There is one caveat. It could be that by moving the boundaries about we had to cross the original point A. Thus, we are guaranteed to obtain a Cauchy surface, but not one with the spacetime point A on it. By varying the Lorentz frame chosen one can presumably avoid losing the point A; this implies that there will be a natural constraint on the set of Lorentz frames allowed near A. Alternatively, we could keep the Cauchy surface we get and then allow it to evolve either towards the future or towards the past such that A is reached again. But this also may imply that the local Lorentz frame at A is altered, if A crosses a boundary.

It is important to see that our procedure will provide us with at least one Cauchy surface that consists of just one polygon, with elaborate matching conditions at its boundaries. It may be indicated by a diagram—see figure 14. The diagram must be read as follows: draw the diagram on an S_2 sphere and then cut it open along the lines. One then obtains the topology of the polygon on the left. Naturally, here we have a universe whose 2-space has the S_2 topology. It should also be evident that this topological structure of the Cauchy surface cannot alter during any of the transitions A–J.

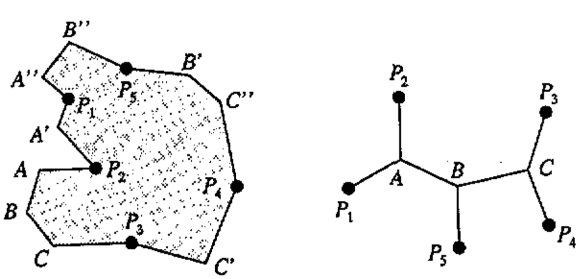

Figure 14. Example of a single-polygon universe and corresponding diagram. P_{1-5} are particles, A, B and C are vacuum vertices.

We are now in a position to calculate the number of independent physical degrees of freedom of an N-particle universe. Let us first choose the Lorentz frame and the time t, and consider a single-polygon Cauchy surface. Assume first the topology to be S_2. How many degrees of freedom does it have? We can choose all η parameters, one for each link, freely. There are $2N-3$ of them (see figure 14, where $N=5$). The triangle relation (2.9) and the mass relation (2.4) then determine all angles of the polygon (the ambiguity in inverting the cosines is lifted by noting that the relative signs of s_i are fixed by (2.5), and by requiring that at each vertex not more than one of the s_i be negative). A constraint is then that the polygon (see figure 14) should close properly. The sum of all angles is fixed. Thus we are left with $2N-4$ freely adjustable parameters. Note now that we have a two-parameter freedom to choose the velocity of the Lorentz frame. Therefore, if we want to know the number of physically observable degrees of freedom we must subtract 2. Thus we finally determine that the total number of all angular and velocity degrees of freedom in an N-particle universe is $2N-6$. It is only the relative velocities that we count. Similarly, only the relative angles are determined. The absolute orientation of our coordinate frame is irrelevant. The same number $2N-6$ can also be obtained by looking at the frame in which one given particle is at rest. In this case the η variable at its wedge should vanish. In addition the angular direction of the wedge is free and therefore not physically relevant, and so we end up with the same number.

Next we consider positions. The lengths of all edges are free, except that two matching edges must be equal in length. Correspondingly we get $2N-3$ parameters, one for each line in the diagram. Again we must require that the polygon closes. All angles and the lengths of all sides have been fixed, so if we travel around the polygon we must check that beginning and end points coincide. This gives two constraints, leaving us with $2N-5$ parameters for the lengths. We conclude that the total number of physically observable parameters is $4N-11$.

What is remarkable about this result is that the number is odd. Usually in dynamical systems we have equal numbers of coordinates and momenta, and therefore the number of freely adjustable degrees of freedom is then even. We have to realize now that we indeed counted one degree of freedom which is actually unobservable. It is the *time* t. Like the coordinates x,y, it depends on the choice of the origin, which is arbitrary, and hence we should not count it as physically observable. If we consider the set of all Cauchy surfaces at all times t and say that this describes one single universe then we see that the number of parameters describing it is $4N-12$.

Next, consider a universe whose 2-space has the topology of a torus, $S_1 \times S_1$. Again we construct a single polygon. Its diagram should be seen as living on a torus, so it is periodic—see figure 15. The number of different edges is now $2N+3$. Again each of these carries an independent parameter η, and together they determine all angles. Subtract

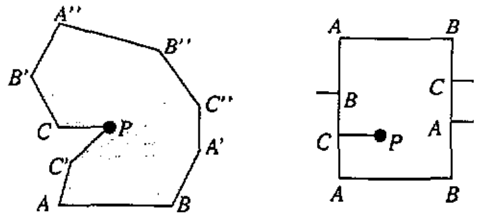

Figure 15. $S_1 \times S_1$ universe with one particle. The diagram is periodic horizontally and vertically.

one degree of freedom for the constraint for the total of all angles, and two for the freedom to choose the Lorentz frame. We find $2N$ momenta. As in the previous case, the $2N+3$ edges each determine a parameter with the dimension of length, after which we impose two constraints such that the polygon closes properly. Thus there are $2N+1$ position parameters. If, as before, we remove the time parameter we see that this universe is determined by $4N$ independent parameters.

It is not difficult to convince oneself that the number n of parameters should be a linear function of the genus g of the universe. We conclude therefore that the number of physical degrees of freedom in a $(2+1)$-dimensional universe of genus g, containing N-point particles, is

$$n = 4N + 12g - 12\,. \tag{4.1}$$

Note that we did not count the masses of the particles, which should be fixed numbers. Furthermore we observed that the genus g cannot change as a function of time. The time coordinate always has topology R_1, except that we have not yet specified whether the allowed time interval runs from $-\infty$ to $+\infty$ or whether there are limits. We will find that more often than not there are boundaries in time.

To choose a sensible initial condition with which one can run a computer simulation is not so easy. To determine the angles (and with those the velocities) one should in principle choose the η parameters for all but one of the edges. This fixes all angles except for one freedom, which then should be determined by noting that all angles in the polygon should add up to a fixed multiple of π (depending on the number of edges). In practice it is easier to fix all η and fix all but one of the masses. By going around the polygon one finds the angle β for the last particle, and together with its η parameter (2.4) then determines its mass. Next, one chooses the lengths of the edges and adjust these such that the polygon closes.

5. The evolution

We are now in a position to switch on our program and see how this universe, with some initial conditions imposed, evolves in time, both towards the future and towards the past. This the author did, on a small machine. Quite a few minor technical problems had to be overcome, most of which are not relevant to our discussion. Let us outline the general procedure. We have a list of all relevant parameters of the universe at a given time t_1. First we use (2.11) and (2.12) to calculate for each edge whether it is a shrinking one, and if so what is the time t_2 at which its length vanishes. Remember which edge vanishes first. Next we consider for all polygons all corners at which the angle exceeds 180° (we call such a corner a concave point). Often but not always this point corresponds to a particle. We have to find out at what time it hits one of the opposite sides. This is technically a bit tricky, but of course the geometry of the problem is completely straightforward. Determine the distances to all edges and the time dependence of these. When one of these distances vanishes we must check whether the concave point actually hits this edge or passes it outside its end points. The need to do this lengthy calculation was the main factor slowing down my program. If a concave point hits one of the other edges of its polygon before any of the edges actually shrink to zero we have to perform transition C or D. Otherwise it is one of the other transitions, for which the rules are also straightforward.

A simulation program that decides to make transition C or D may well make an error that requires some skill to correct. A point one could easily overlook is that during the evolution

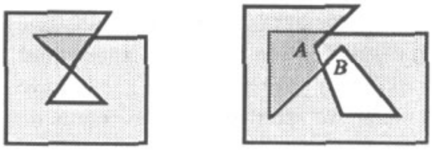

Figure 16. Self-overlapping polygon. In the polygon at right the concave points A and B may hit one of the other edges and yet no transition of type C or D should occur.

as fixed by the rules of section 2 a polygon, if described in locally Cartesian coordinates, may become self-overlapping. This is because there are concave corner points—see figure 16. The coordinates of one of the concave corner points (point A or B in figure 16) may well coincide with one of the other edges, but it may live in another Riemann sheet. In that case the transition of type C or D should be over-ruled.

Naturally, one can teach a computer program to recognize this case by considering the total winding number of the angles of the edges along the path from the concave point to the edge hit. In practice, however, this circumstance is so rare that one may decide to ignore it, but every now and then the program will crash. We did observe this happening on several occasions. Self-overlapping polygons do occur more often, but in most cases there is no concave point hitting another edge (figure 16 (left)).

Studying the transitions while they take place turned out to be instructive. We took for instance the universe depicted in figure 14 (the sketch is not rendered with much precision there, but it does represent a large class of cases studied). The three vacuum vertices A, B and C are only surrounded by convex angles, and this implies that all velocities have the same sign. Suppose the signs are chosen such that all polygons shrink. A theorem derived in [3] then states that the shrinking will go on forever. The theorem is easily proved by considering all possible transitions one by one. If all angles are convex (apart from the wedges attached to the particles) then one easily sees that this situation has to stay that way. We observed that not only the velocities keep their signs, but their absolute values also increase. Thus, the shrinking is accelerated. After a limited number of transitions one recognizes a pattern that will persist, and is described in the next section. The transitions continue to take place, but the shrinking goes so rapidly that the Cauchy time parameter does not exceed a limiting value.

Let us now consider the case where the universe starts out expanding, or alternatively, let us consider extrapolating the previous case backwards in time. One could expect this expansion also to last forever, and indeed, if we choose our initial state such that the polygon inflates more or less conformally, i.e. keeping its original shape, then not only do all edges increase, but no concave corner point will ever hit an opposite edge, and so we are in a final state.

However, very often this is not at all what happens. If we take the expanding case of figure 14 we see that first particles P_1 and P_2 will come close to each other, and a transition may take place that creates concave angles and a negative velocity. The time-reverse of the just-mentioned theorem now tells us that the situation with *at least one* of the velocities negative must also last for ever. Whether or not this is sufficient to terminate the expansion now depends delicately on the initial conditions. Often hundreds of transitions take place, and along the way the universe may have split into dozens of polygons. We found that in many cases more velocities with negative signs show up, until indeed the whole thing begins to shrink. The asymptotic state of the eternally shrinking universe is again reached (although sometimes one of the velocities may stay positive). What we have found in this way is a universe that has a finite timespan. It began its existence at time $t = t_0$ and terminated it with a crunch at time $t = t_1$.

Curiously, this behaviour seems to be related to the spacelike topology. This was discovered by testing a number of toroidal universes ($g = 1$). Consider figure 15. Since the particle(s) is (are) surrounded by a wedge we get a positive contribution to the Euler characteristic of the surface from the particle(s). Since the integrated curvature of the Cauchy surface must now vanish we must ask for a negative contribution from the total of all vacuum vertices. They contribute negatively if all velocities η have the same sign. Consequently the toroidal universes tend to fall into two classes, those where the majority of all velocities are positive and those for which the majority of the velocities are negative. In spite of many trial runs we were unable to find any example of a transition from one class into the other. In other words, each of these universes is either always expanding or always shrinking; the time axis is always a half-line. We now conjecture that this is a general rule for $g = 1$ universes. Furthermore we guess that if $g > 1$ the time axis may run from $-\infty$ to $+\infty$, because such a universe might grow to infinite size both in the future and in the past. Note that this conjectured relation between spacelike curvature and timelike compactness is the same as in $(3+1)$-dimensional Friedmann universes.

6. The asymptotic state

The asymptotic state for an expanding universe is very simple. Since the velocities will be constant one will end up with a polygon or a set of polygons whose shapes and relative sizes become constant while their sizes simply scale as a function of time. In contrast, the shrinking universe requires much more discussion.

Regardless of how many polygons the Cauchy surface has developed during its evolution, we find the number to decrease again, and the shrinking universe always ends up in just one polygon. It usually approaches the shape of a triangle; other cases, which we will discuss shortly, can be reduced to this one by Lorentz transformation, so let us discuss this simplest case first. Suppose we have a single polygon approximately of the form sketched in figure 17. The η parameters at the edges are all large and the signs are such that the edges move inwards. They move very nearly with the speed of light. The points A′, A″ are image points of A and similarly B and C have image points. The particles P_1–P_5 also move nearly with the speed of light inwards, and since they stay on the edges the direction of their movements is pracitcally orthogonal to the edges. The angles $2\beta_i$ of the wedges, such as A–P_1–A″ have become nearly 180° because of the Lorentz boosts. The diagram for the configuration drawn is as the one in figure 14. Note that the image points as well as the corners move faster than light along trajectories which are not orthogonal to the edges. They are of course allowed to do so, being artefacts of our coordinate frame.

How will this configuration evolve? The edges move inwards, the particles move on lines orthogonal to the edges. Every now and then either a particle or one of the image

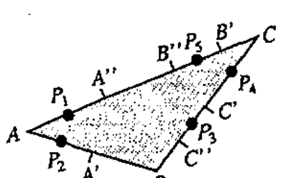

Figure 17. Example of an asymptotic configuration.

points slips off at one end-point of the edge it is on. It will reappear at one of the two image points of this end-point, all according to transition A or transition B of section 2 (the other transitions will not occur any more). In the case of transition B it is easy to see that the particle at its new position will have been boosted towards still higher energy. But also in case of transition A the newly opened edge will have an even higher velocity than the previous one. In both cases the new boost parameter η_1 is determined by the old parameters η_2 and η_3 *via* (2.9), where c_1 is the cosine of the angle at the corner. For simplicity let us define the sign of η such that it is positive for a shrinking universe. For large η we have

$$\gamma = \sigma = \tfrac{1}{2}e^{2\eta} \tag{6.1}$$

so that

$$e^{2\eta_1} = e^{2\eta_2}e^{2\eta_3}\,\tfrac{1}{2}(1 + c_1)\,. \tag{6.2}$$

Even if c_1 is close to -1 this will give

$$\eta_1 \gg \eta_2\,,\ \eta_3 \tag{6.3}$$

as soon as all η are sufficiently large.

The new angles will be fixed by (2.10) which will become

$$\cot\alpha_2 = -\frac{1 + c_1}{2s_1}e^{2\eta_3}\,. \tag{6.4}$$

Since this is negative and large the new angles will all be close to 180°.

Thus we see that after any of the transitions the polygon will be even closer to a pure triangle than before. All corners except A, B and C will be very close to 180° and the edges will move with velocities rapidly approaching that of light.

How fast will the speed of light be reached? We see from (6.2) that whenever the triangle scales to a fraction ρ of its previous size the parameters η scale by a similar factor. We derive from that

$$\eta(t) \to (t_0 - t)^{-\kappa} \tag{6.5}$$

$$\gamma(t) \to \exp[2(t_0 - t)^{-\kappa}] \tag{6.6}$$

where κ is some critical coefficient which we have not yet determined.

Occasionally we find an apparently different asymptotic behaviour. The final polygon then has the shape of a quadrangle, much like the 'kite' in figure 12. At the concave angle there is one slowly moving particle. The two adjacent sides are the wedge of this particle and hence are equal in length. Just because the particle moves these two edges can stay of equal length even if the quadragle is not reflection-symmetric. All other particles and image points are on the two other sides, which will move with nearly the speed of light. Since the wedge edges must be of equal length at all times no particle or image point can ever appear there. Arguments similar to the ones for the previous case show that this configuration, while scaling to ever decreasing size, is stable.

Actually there is no physical distinction between these two possible asymptotic universes. One is simply a Lorentz transform of the other. In fact, one can prove that by performing Lorentz transformations one can transform the triangle of figure 17 into any other convex triangle. So its shape is not a physical characteristic of this universe but only of our Cauchy surface. Our construction procedure is such that one cannot foresee which set of Cauchy surfaces one will end up with, but for the description of what is going on this does not matter much.

Thus we can just as well first transform towards an equilateral triangle to describe the generic asymptotic state. It is easy to see that the order at which the transitions will occur will become infinitely sensitive to the initial conditions. Rather than seeing just N particles coming towards him an observer in this universe will see three 'walls' coming towards him that are infinitely dense with particles. This is because every particle has an infinite series of image points all situated on or very near the edges.

In every respect the crunch at $t = t_0$ is a veritable crunch. The particles come closer and closer together, missing each other at ever decreasing impact parameters. There is no timelike trajectory that is not approached arbitrarily closely by an arbitrarily large number of particles in a time interval close enough to t_0, as seen in its own Lorentz frame. There is no way to extend this spacetime analytically beyond that time t_0.

7. Discussion and conclusion

There are two possible classes of asymptotic states, a crunch (C) and an infinite expansion (E). The infinite expansion is relatively simple to describe, since the universe, including its particles, will keep a fixed shape and just scale towards infinite sizes. The velocity is fixed and less than that of light, so the expansion will last for ever. In contrast, the crunch is basically chaotic. The arrangement of particles as seen by a local observer changes continuously, depending on an ever increasing number of decimal places in the parameters describing the initial condition.

At earlier stages of the crunch one may find instead of figure 17 a polygon with more than three sides, or more than one polygon. But in the generic case such configurations do not shrink evenly. One edge shrinks to zero before the others do, or one polygon disappears earlier than the other. What is left then is invariably figure 17 (with arbitrary distributions of particles and image points on the edges).

These asymptotic states describe the limiting situation both in the far future and in the far past. Backwards in time the crunch corresponds to a 'big bang' (but we will keep the letter C to indicate such a past history). Using simple computer simulation techniques we ran a large number of examples and found that if the topology of the two-dimensional Cauchy surfaces is chosen to be S_2 then the following three universe types are possible: (i) E–C (ii) C–E (iii) C–C. In the first two cases time is a half-line. In case (iii) time is a finite interval.

If 2-space has an $S_1 \times S_1$ topology we could only find two cases: E–C and C–E. Extrapolating this we conjecture that for $g > 1$ universes the configuration E–E is also possible, but again not C–C.

The intermediate phase between the two asymptotic situations can be very complex, and it may easily take hundreds of transitions, during which dozens of polygons may be formed. Because of the chaotic nature of the crunched state it will be impossible to foresee whether the universe will be of type (ii) or type (iii) when we start off with a sufficiently 'crunched' configuration.

An important motivation for this study was the hope to get a more complete understanding of possible quantum versions of this system. (Our work suggests as a natural choice of canonical variables the Lorentz boost parameters η_i and the lengths L_i of the edges i which, however, are subjected to constraints.) Ideally, a quantum theory would give us a matrix of transition amplitudes between asymptotic states at the beginning and asymptotic states at the end of the universe:

$$\langle \text{out} | \text{in} \rangle \,. \tag{7.1}$$

Now we see that the asymptotic states form two widely different classes. If we have an asymptotically expanding state we may have some intuition on how to replace classical particles by wavepackets [4], although we have to remember that at each instant in time there are only $4N - 11$ classical degrees of freedom (in the spherical case), much less than the usual $4N$, and on top of that an odd number. The asymptotic state is presumably better described by first eliminating the 'irrelevant' time coordinate t.

But how should we define the asymptotic states in the case of a crunch? Perhaps they are again semiclassical. Remember that the distances Δx decrease as a linear function of $t_0 - t$, whereas the 'energies' γ_i and 'momenta' $p_i = \sigma_i$ grow exponentially as a power of $1/(t_0 - t)$. So in the limit $t \to t_0$ we might have wavepackets where the effects of the uncertainties,

$$\Delta x \Delta p \tag{7.2}$$

become unimportant.

However, a proper formulation of a quantum theory, invariant under the transitions A–J as described in section 2, seems to be a long way off. It is also not at all clear to this author whether a (not too trivial) finite-dimensional quantum version of this theory should exist at all. Note that the theories described in [5] refer to the infinite-dimensional field theory *before* the constraint that spacetime is locally flat has been exploited. These theories also allow for the possibility of topology change, which in our classical version is obviously forbidden. It is also important to realize that also in $2 + 1$ dimensions quantum gravity has a natural unit of length, suggesting that both space and time are quantized. What this will mean in practice is still a mystery.

References

[1] Staruszkiewicz A 1963 *Acta Phys. Polon.* **24** 734
Gott J R and Alpert M 1984 *Gen. Rel. Grav.* **16** 243
Giddings S, Abbot J and Kuchar K 1984 *Gen. Rel. Grav.* **16** 751
Deser S, Jackiw R and 't Hooft G 1984 *Ann. Phys.* **152** 220

[2] Gott J R 1991 *Phys. Rev. Lett.* **66** 1126

[3] 't Hooft G 1992 *Class. Quantum grav.* **9** 1335

[4] 't Hooft G 1988 *Commun. Math. Phys.* **117** 685

[5] Witten E 1988 *Nucl. Phys.* B **311** 46
Carlip S 1989 *Nucl. Phys.* B **324** 106