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Some notes on the graded rotation group

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Abstract

Rotations in 3-dimensional space can be complemented by *graded rotations*. The 3 angular momentum operators receive two complex superpartners. The irreducible representations of this graded $SU(2)$ symmetry group each consist of one vector with angular momentum $\ell = \ell_G$ and one with angular momentum $\ell = \ell_G - 1/2$, so that the dimension of such a representation is $4\ell_G + 1$. Unlike conventional supersymmetry, here the fermionic representation has either one component more or one component less than the bosonic one. The case $\ell_G = 0$ is the trivial representation. It seems that this graded algebra only allows $N = 1$ supersymmetry, not the higher N values.

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1. Introduction

Usually, supersymmetry is regarded as a relation between bosons and fermions in a quantum field theoretical setting. The fact that the anticommutator between two supersymmetry generators is proportional to the momentum operators, the generators of translations, appears to be a curiosity needed to relate the fermionic and bosonic field equations, rather than a special feature of the translation group itself. The generator of time translations, the Hamiltonian, is conveniently written as the anticommutator of an operator with itself, *i.e.*, the square of that operator.

Alternatively,one may view supersummetry as the super extension of the translation group, an important property of space-time. The fact that one can do the same thing with the rotation group seems to be less evident, and in any case it is rarely mentioned.

The nice thing about the rotation group is that it has a denumerable set of finite-dimensional representations, which are particularly simple in the three-dimensional case. Its graded extension is also simple, and here we review the exercise that one can do. This author had been unaware of these nice features, so he decided to write this down. I did not find this simple discussion in the literature. It seems that only the $N = 1$ superextension gives finite-dimensional representations, as will be demonstrated.

2. The algebra

The $SU(2)$ rotation group has the generators L_a , $a = 1, 2, 3$, from which we will use L_3 and $L_\pm$, where $L_pm = L_1 \pm iL_2$. The commutation rules are

$$[L_3,\, L_\pm] = \pm L_\pm \ ; \qquad [L_+,\, L_-] = 2L_3 \ . \tag{2.1}$$

The Casimir operator is

$$\sum_a L_a^2 \ = \ \ell(\ell + 1) \ , \tag{2.2}$$

where ℓ is an integer or an integer plus $\frac{1}{2}$. At each of these values of ℓ , there is exactly one irreducible representation with dimensionality $2\ell + 1$. The states are indicated as $|\ell,\, m\rangle$, with $-\ell \le m \le \ell$. A convenient representation of the three operators in these representations is

$$L_\pm|\ell,\, m\rangle \ = \ \sqrt{(\ell \mp m)(\ell + 1 \pm m)}\ |\ell,\, m \pm 1\rangle \tag{2.3}$$

$$L_3|\ell,\, m\rangle \ = \ m\ |\ell,\, m\rangle \ . \tag{2.4}$$

We now search for anticommuting generators, such that their anticommutators generate the angular momentum operators L_a . After a little bit of thought, one finds that the minimal number of such operators is likely to be two, which we call ψ_1 and ψ_2 , forming the complex two-representation of the rotation group. Thus they should commute with the angular momenta as follows:

$$[L_3,\,\psi_1] = -\tfrac{1}{2}\psi_1\ ,\qquad [L_+,\,\psi_1] = \psi_2\ ,\quad [L_-,\,\psi_1] = 0\ \ ; \tag{2.5}$$
$$[L_3,\,\psi_2] = \ \tfrac{1}{2}\psi_2\ ,\qquad [L_+,\,\psi_2] = 0\ ,\quad [L_-,\,\psi_2] = \psi_1\ . \tag{2.6}$$

The fact that this algebra is consistent up to this point is fairly obvious and can be easily checked. The more subtle part of the algebra will be the relation between the anticommutators and the $L|a$ operators. The following is consistent, and up to trivial rescalings, it is unique:

$$\{\psi_1,\,\psi_1\} = L_-\ ,\quad \{\psi_2,\,\psi_2\} = -L_+\ ,\quad \{\psi_1,\,\psi_2\} = L_3\ . \tag{2.7}$$

Careful checks show that this algebra closes correctly. However, even if the algebra closes, it still might not have finite-dimensional representations. This algebra turns out to be completely acceptable, as the following section shows.

3. Representations

We proceed to identify the complete set of representations. The operator (2.2) commutes with all L_a, but not with the ψ_i, and so it is not a Casimir operator for the graded group. One finds

$$[\vec{L}^2,\,\psi_1] = \psi_2 L_- - \psi_1 L_3 + {}^3\!/_4\psi_1 = L_-\psi_2 - L_3\psi_1 - {}^3\!/_4\psi_1\ . \tag{3.1}$$

To complete the Casimir operator, we need a combination of ψ's that is also a scalar under ordinary rotations. A candidate is

$$Q = \psi_1\psi_2 - \psi_2\psi_1\ . \tag{3.2}$$

It indeed commutes with all L's. Its commutator with ψ_1 is

$$[Q,\,\psi_1] = 2L_3\psi_1 - 2L_-\psi_2 + {}^3\!/_2\psi_1 = 2\psi_1 L_3 - 2\psi_2 L_- - {}^3\!/_2\psi_1\ . \tag{3.3}$$

So, indeed, the operator

$$R = \vec{L}^2 + \tfrac{1}{2}Q \tag{3.4}$$

is the Casimir operator of the above algebra.

This information should be enough to generate the representations. But one further algebraic relation is essential. By elaborating the square of Q, one finds

$$Q(Q-1) = \ell(\ell+1)\ . \tag{3.5}$$

This equation can be solved for Q:

$$Q = \ell + 1 \quad \text{or} \quad -\ell\ , \tag{3.6}$$

which we rewrite as

$$Q = \tfrac{1}{2} + \sigma(\ell + \tfrac{1}{2}) \ , \quad \sigma = \pm 1 \ . \tag{3.7}$$

In terms of this operator, the Casimir operator (3.4) becomes

$$R = (\ell + \tfrac{1}{2})(\ell + \tfrac{1}{2} + \tfrac{1}{2}\sigma) \ . \tag{3.8}$$

Since σ can only take two values, and R must be the same for all states in one irreducible representation, we find that the two states with $\sigma = \pm 1$ differ exactly half a unit of ℓ. If the state with $\sigma = -1$ has total angular momentum ℓ_G, the Casimir operator is $R = (\ell_G + \frac{1}{2})\ell_G$. The state with $\sigma = +1$ must have the same value of R, and therefore, it has angular momentum $\ell = \ell_G - \frac{1}{2}$. Thus, we established the nature of the representations. We call them $[[\ell_G]]$, where the number ℓ_G can be an integer or an integer plus $\frac{1}{2}$. Under $SU(2)$, the representation $[[\ell_G]]$ splits up as

$$[[\ell_G]] \ = \ \{[\ell_G] \oplus [\ell_G - \tfrac{1}{2}]\} \ , \tag{3.9}$$

where the ordinary $SU(2)$ representations with angular momentum ℓ are indicated as $[\ell]$. The dimensionality of these representations is $2\ell_G + 1 + 2\ell_G = 4\ell_G + 1$. Note that $\ell_G = 0$ here is also a representation because, in this case, the $\ell = \ell_G - \frac{1}{2}$ component disappears. It is the trivial representation for which $L_a = 0, \ \psi_i = 0$.

4. The matrix elements

We have the explicit matrix elements for $L_\pm$ and L_3, as given in Eqs (2.3) and (2.4). What are the matrix elements of the ψ_i? We know that ψ_1 subtracts, and ψ_2 adds one-half to L_3, so they must flip the operator σ. Writing an ansatz for these operators, and plugging them into the commutator relations (2.7), one readily finds (writing ℓ instead of ℓ_G for short):

$$\psi_1 \, |\ell, m\rangle \ = \ \sqrt{\tfrac{1}{2}(\ell + m)} \ |\ell - \tfrac{1}{2}, \, m - \tfrac{1}{2}\rangle \ , \tag{4.1}$$

$$\psi_1 \, |\ell - \tfrac{1}{2}, \, m + \tfrac{1}{2}\rangle \ = \ \sqrt{\tfrac{1}{2}(\ell - m)} \ |\ell, m\rangle \ , \tag{4.2}$$

$$\psi_2 \, |\ell, m\rangle \ = \ -\sqrt{\tfrac{1}{2}(\ell - m)} \ |\ell - \tfrac{1}{2}, \, m + \tfrac{1}{2}\rangle \ , \tag{4.3}$$

$$\psi_2 \, |\ell - \tfrac{1}{2}, \, m - \tfrac{1}{2}\rangle \ = \ \sqrt{\tfrac{1}{2}(\ell + m)} \ |\ell, m\rangle \ . \tag{4.4}$$

Note that, in these expressions, ℓ keeps the same value throughout the representation R_ℓ. The quantum number m actually suffices to identify the basis element of the representation; it takes the $4\ell + 1$ values $-\ell, \ -\ell + \frac{1}{2}, \ \cdots, \ \ell - \frac{1}{2}, \ \ell$.

5. The question of higher N algebras

One may think of generalizing the algebra by taking $2N$ anticommuting generators, ψ_i^p, with $i = 1, 2$ and $p = 1, \cdots, N$. Let the commutators (2.5) and (2.6) hold for each p separately. One might suspect that the equations (2.7) could be replaced by

$$\{\psi_1^p, \psi_1^q\} = L_- \delta^{pq} \ , \quad \{\psi_2^p, \psi_2^q\} = -L_+ \delta^{pq} \ , \quad \{\psi_1^p, \psi_2^q\} = L_3 \delta^{pq} \ , \tag{5.1}$$

in analogy with conventional supersymmetry. One could define

$$Q^p = \psi_1^p \Psi_2^p - \psi_2^p \psi_1^p \ , \tag{5.2}$$

which, besides the same commutator (3.3) with ψ_1^p, would commute with the ψ_i^q, if $q \neq p$. The Casimir operator would be

$$R = \vec{L}^2 + \tfrac{1}{2} \sum_p Q^p \ , \tag{5.3}$$

and since Eq. (3.5) would hold for every p separately, we would have

$$Q^p = \tfrac{1}{2} + \sigma^p (\ell + \tfrac{1}{2}) \ , \quad \sigma^p = \pm 1 \ . \tag{5.4}$$

This Casimir operator, Eq. (5.3), would be worked out to be

$$R = (\ell + \tfrac{1}{2})(\ell + \tfrac{1}{2} + \tfrac{1}{2} \sum_p \sigma^p) + \tfrac{1}{4}(N - 1) \ , \tag{5.5}$$

but, since $\sum \sigma^p$ can now take more than two values, the argument of Section 3 fails; the various ℓ values differ by numbers that are not half of an integer. Hence, we do not have finite-dimensional non-trivial representations of this algebra.

This was clearly not an exhaustive search for higher N supersymmetric generalizations of the angular momenta. Presumably, besides the $SU(2)$ doublets, one will have to use higher spin fermionic generators.

6. Discussion

The symmetry discussed here differs from conventional supersymmetry. It puts exactly one object with angular momentum $\ell = \ell_G$ and one with angular momentum $\ell - \ell_G - \frac{1}{2}$ in one supermultiplet. Thus, the number of "fermionic" components differs from the number of "bosonic" ones by plus or minus one unit. One might allpy this to conventional rotations, but we could also consider an internal symmetry group such as isospin. In that case, the Lorentz spin stays unaffected, so that we would not be dealing with a symmetry that transforms bosons into fermions and *vice versa.*