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RENORMALIZATION OF MASSLESS
YANG-MILLS FIELDS

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Abstract: The problem of renormalization of gauge fields is studied. It is observed that the use
of non-gauge invariant regulator fields is not excluded provided that in the limit of high
regulator mass gauge invariance can be restored by means of a finite number of counter-
terms in the Lagrangian. Massless Yang-Mills fields can be treated in this manner, and
appear to be renormalizable in the usual sense.
    Consistency of the method is proved for diagrams with non-overlapping divergencies
by means of gauge invariant regulators, which however, cannot be interpreted in terms of
regulator fields. Assuming consistency the S-matrix is shown to be unitary in any order
of the coupling constant. A restriction must be made: no local, parity-changing transfor-
mations must be contained in the underlying gauge group. The interactions must con-
serve parity.

1. INTRODUCTION

In recent years the Feynman rules for massless Yang-Mills fields have been es-
tablished [1-5]. Naive power counting suggests a renormalizable theory; however,
in order to carry through a renormalization procedure one must first define a cut-off
procedure. And if the cut-off procedure breaks the gauge-invariance of the theory
then it is no more clear what the Feynman rules are. The reason is that gauge-
invariance, through Ward identities, is essential for the S-matrix to be unitary.

Thus the problem poses itself as follows: how to find a gauge invariant cut-off
procedure. This problem is of course quite the same in quantum electrodynamics.
There the problem was solved by Pauli, Villars [6] and Gupta [7] who succeeded
in finding a set of regulator fields that could be coupled in a gauge invariant way.
Now in the case of massless Yang-Mills fields a gauge invariant regularizing proce-
dure also seems to exist. Unfortunately, however, this procedure cannot be inter-
preted in terms of fields with indefinite metric and/or wrong statistics, like in the
case of electrodynamics. Hence, unitarity and causality are no longer evident.

However, it must be realized that the whole renormalization procedure involves

also the addition of counterterms in the Lagrangian. And in fact the important point is that the total effect of regulator fields and counterterms is to be gauge invariant, at least in the limit of infinite regulator masses. Thus let us suppose now that we have found a set of regulator fields, that makes the various amplitudes finite but destroys the gauge invariance. If we are to restore gauge invariance by means of a finite number of counterterms in the Lagrangian then the gauge-invariance breaking terms in the above mentioned amplitudes must be polynomials of a definite degree in the external momenta, order by order in perturbation theory. But this is precisely the same problem as with the ultra-violet infinities in perturbation theory: the cut-off dependent terms must be polynomials of a definite degree in order for the theory to be renormalizable. Thus the usual proofs of the renormalizability of quantum electrodynamics also guarantee that the unwanted effects of a non-gauge invariant regulator procedure may be off-set by suitably chosen counterterms. Our aim with this procedure is twofold: first, causality is evident, and unitarity can be proven using Cutkosky relations. Secondly, actual calculations are much easier this way, because the counterterms can be fixed easily by applying Ward identities, whereas gauge-invariant regulators become rather complicated particularly at higher orders.

The above point may be illustrated in quantum electrodynamics; in sect. 2 our cut-off procedure is applied to the lowest order photon self energy diagram. Here the unwanted effects of a non-gauge invariant regulator procedure are seen to be such that they can be cured by means of counterterms, one of which has the form of a photon mass term.

One may argue that the method is equivalent with a dispersion relation technique, where the subtraction constants are determined by generalized Ward identities; and that is then sufficient to have a completely gauge invariant theory.

In sect. 3 the situation for massless Yang-Mills fields is outlined. First we use non-gauge invariant regulators, and require that counterterms that remove divergencies are such that Ward identities hold*.

Consequently, three important questions must be answered:

(i) Do the Ward identities determine the hitherto arbitrary coefficients uniquely? Indeed, we will show that only one arbitrary physical constant remains, being the renormalized coupling constant. Two other arbitrary numbers are unobservable and can be chosen by some convention.

(ii) Are there no internal inconsistencies, like in the PCAC case [8, 9], where no renormalizable counterterm could be found in such a way that PCAC and gauge invariance hold at the same time? In sect. 4 we show a combinatorial proof of the Ward identities, and it appears that many shifts of integration variables are necessary for this proof. Nevertheless, there are no inconsistencies, and for the case of one closed loop we prove this by deriving the gauge invariant set of regulators already referred to (sect. 5). Extension of a similar regulator technique to higher-orders

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* The method of removing infinities by the use of Ward identities and counterterms for Quantum Electrodynamics is described in Jauch and Rohrlich, Theory of photons and electrons, p. 189.

seems possible, but complicated and tricky, and we shall not bother about it in this article.

(iii) Is the resulting S-matrix unitary? In sect. 5 we generalize the Ward identities, in order to show that the ghost particle intermediate states cancel the intermediate states with non-physically polarized W-particles. Thus in the unitarity equation only physically (i.e. transversely) polarized W-particles occur in the intermediate states.

In appendix A a simple formal path integral derivation of the Feynman rules for Yang-Mills fields and the generalized Ward identities is given for both Landau and Feynman gauge. The rules are listed in appendix B.

We use the notation $k_\mu = (k, ik_0); k^2 = k^2 - k_0^2$. Throughout the paper we confine ourselves to the perturbation expansion. The underlying group here is SU(2), though this is not essential. For simplicity also, no other particles with isospin are taken into account, but introducing them does not give rise to any serious difficulty, as long as the matrix γ^5 and the tensor $\epsilon_{\kappa\lambda\mu\nu}$ do not occur in the Lagrangian.

2. QUANTUM ELECTRODYNAMICS

In this section we review the situation in quantum electrodynamics. We calculate the contribution of the diagram in fig. 1 to the photon self-energy: a spin $\frac{1}{2}$ particle forms a closed loop. We do this calculation in order to show the procedure, which can readily be extended to non-Abelian gauge fields.
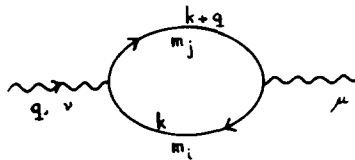

Fig. 1.

The integral diverges quadratically. Now suppose we regularize by replacing the propagator $(m + i\gamma k)^{-1}$ by

$$\sum_i c_i (m_i + i\gamma k)^{-1} ,$$ (2.1)

with

$$\sum_i c_i = 0 , \qquad \sum_i c_i m_i = 0 , \qquad \sum_i c_i m_i^2 = 0 , \qquad c_0 = 1 , \qquad m_0 = m , \quad (2.2)$$

and let ultimately m_i go to infinity for $i \neq 0$ (c_i remain finite).

For finite m_i the integral now converges and we may shift the integration variable and integrate symmetrically. Then we have

$$
\Pi_{\mu\nu} = -\frac{ie^2}{(2\pi)^4} \int d^4k \sum_{ij} c_i c_j \frac{\mathrm{Tr}(m_i - i\gamma k)\gamma_\mu (m_j - i\gamma(k+q))\gamma_\nu}{(k^2 + m_i^2)((k+q)^2 + m_j^2)}
$$

$$
= -\frac{4ie^2}{(2\pi)^4} \int_0^1 dx \int d^4k \sum_{ij} c_i c_j \frac{(m_i m_j + \frac{1}{2}k^2 - x(1-x)q^2)\delta_{\mu\nu} + 2x(1-x)q_\mu q_\nu}{[k^2 + m_i^2 x + m_j^2(1-x) + q^2 x(1-x)]^2} .
$$

$$(2.3)$$

Let us define

$$
\mu_{ij}^2 \equiv m_i^2 x + m_j^2(1-x) + q^2 x(1-x) ,
$$

$$(2.4)$$

then we also have

$$
\sum_{ij} c_i c_j \mu_{ij}^2 = 0 ,
$$

$$(2.5)$$

and we can evaluate the convergent integral using

$$
\int \sum_{ij} c_i c_j \frac{d^4k}{(k^2 + \mu_{ij}^2)^2} = -i\pi^2 \sum_{ij} c_i c_j \log \mu_{ij}^2 ,
$$

$$
\int \sum_{ij} c_i c_j \frac{m_i m_j \, d^4k}{(k^2 + \mu_{ij}^2)^2} = -i\pi^2 \sum_{ij} c_i c_j m_i m_j \log \mu_{ij}^2 ,
$$

$$
\int \sum_{ij} c_i c_j \frac{k^2 \, d^4k}{(k^2 + \mu_{ij}^2)^2} = 2i\pi^2 \sum_{ij} c_i c_j \mu_{ij}^2 \log \mu_{ij}^2 ,
$$

$$(2.6)$$

so that (2.3) becomes

$$
\left(\frac{e}{2\pi}\right)^2 \int_0^1 dx \sum_{ij} c_i c_j \{\delta_{\mu\nu}(2x(1-x)q^2 + m_i^2 x + m_j^2(1-x) - m_i m_j)
$$

$$
- 2x(1-x)q_\mu q_\nu\} \log [m_i^2 x + m_j^2(1-x) + q^2 x(1-x)] .
$$

$$(2.7)$$

To see what happens if for $i \neq 0$ m_i goes to infinity while the c_i remain finite, we

split off the term $i = j = 0$ and ignore contributions of order q^2/m_i^2 for $i \neq 0$:

$$
\Pi_{\mu\nu} = \left(\frac{e}{2\pi}\right)^2 \int_0^1 dx \left\{ 2x(1-x)(q^2\delta_{\mu\nu} - q_\mu q_\nu) \left[\log(m^2 + q^2 x(1-x)) \right. \right.
$$

$$
\left. + \sum_{ij}{}' c_i c_j \log(m_i^2 x + m_j^2(1-x)) \right]
$$

$$
+ \sum_{ij}{}' c_i c_j \delta_{\mu\nu}(m_i^2 x + m_j^2(1-x) - m_i m_j) \left[\log(m_i^2 x + m_j^2(1-x)) \right.
$$

$$
\left. \left. + \frac{q^2 x(1-x)}{m_i^2 x + m_j^2(1-x)} \right] + \text{terms } O\left(\frac{q^2}{m_{i\neq0}^2}\right) \right\} , \tag{2.8}
$$

where Σ'_{ij} denotes the sum over all i and j except the term with both $i = j = 0$. This result does not satisfy the usual gauge condition

$$
q_\mu \Pi_{\mu\nu}(q) = 0 , \tag{2.9}
$$

and the renormalized mass of the photon is not evidently zero.

Of course, the reason is that our regulators are not gauge invariant; a vertex where a photon line is attached to particle lines with different masses is not allowed. If we had used Pauli-Vilars-Gupta regulator fields instead of the propagators (2.1), that is, if in formulae (2.3)–(2.8) $\Sigma_{ij} c_i c_j$ is replaced by

$$
\sum_{ij} c_i \delta_{ij} ,
$$

then the second term in (2.8) would vanish identically and eq. (2.9) would be fulfilled [6, 7].

However, it is important to note that the gauge non-invariant term in (2.8) is only a polynomial of rank one as a function of q^2. Let us abbreviate it by

$$
\left(\frac{e}{2\pi}\right)^2 (M + L q^2)\delta_{\mu\nu} . \tag{2.10}
$$

It can be removed from expression (2.8) if we add a simple counterterm into the Lagrangian*

$$
\Delta \mathcal{L} = -\tfrac{1}{2} \left(\frac{e}{2\pi}\right)^2 (MA_\mu^2 + L(\partial_\nu A_\mu)^2) . \tag{2.11}
$$

---

* This implies that terms in the Lagrangian are renormalized, not the fields, as is often done.
 The difference is merely a scale transformation of the bare quantities.

These terms are local and have dimension less than or equal to four, so that causality and renormalizability are not destroyed.

This can be seen to be a very general feature: instead of the gauge invariant Pauli-Villars-Gupta regularization technique we could just as well regularize with the revised propagator (2.1) (which is a non-gauge invariant procedure) and add to the Lagrangian as many local counterterms with dimension less than or equal to four, as desirable. All arbitrary coefficients can then be fixed by requiring the validity of identities like (2.9).

Equations like (2.9) will be called generalized Ward identities* from now on. They are derived from the usual Ward-Takahashi identity

$$(p' - p)_\mu \, \Gamma_\mu(p', p) = S_F'^{-1}(p') - S_F'^{-1}(p) \, , \tag{2.12}$$

which can be symbolized as
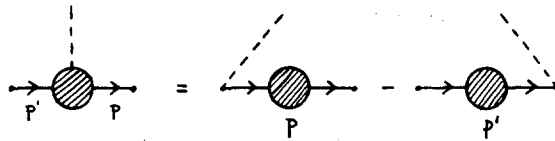
Here the dashed line denotes a "scalar photon" (a photon line with polarization vector proportional to its own momentum). This identity can be used to derive other equalities for diagrams. For instance
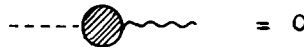
which is precisely eq. (2.9).

In our example we see that the coefficient in front of $(q^2\delta_{\mu\nu} - q_\mu q_\nu)$ is still unspecified. This is because we can add freely counter terms proportional to $F_{\mu\nu}F_{\mu\nu}$ to the Lagrangian because they are gauge invariant themselves. It corresponds to a scale transformation in our definition of the field A_μ. So the freedom we have is only freedom in definition. The most convenient choice is to keep the matrix element of $A_\mu(x)$ between the vacuum and the one-photon state fixed:

$$\langle 0 | A_\mu(x) | k, \epsilon \rangle = \epsilon_\mu \, e^{ikx} \, . \tag{2.13}$$

The renormalized propagator must then have a pole with residue unity at $k^2 = 0$, just as the bare propagator.

* See e.g. J.D.Bjorken and S.D.Drell, Relativistic quantum fields.

So (2.8) must vanish on the mass shell:

$$\Pi_{\mu\nu}(q^2 = 0) = 0 , \tag{2.14}$$

and we derive finally

$$\Pi_{\mu\nu}(q^2) = \left(\frac{e}{2\pi}\right)^2 \int\limits_0^1 \mathrm{d}x\, 2x(1-x)(q^2\delta_{\mu\nu} - q_\mu q_\nu)$$

$$\times \left[\log\left(m^2 + q^2 x(1-x)\right) - \log m^2\right] . \tag{2.15}$$

Once we know that the above mentioned procedure works well, we can go even further and leave the particular set of regulator fields or propagators altogether unspecified. Instead of the identities (2.6) we may use the symbolic expressions:

$$\int \frac{\mathrm{d}^4 k}{(k^2 + \mu^2)^2} = -i\pi^2 \log\mu^2 + D_1 ,$$

$$\int \frac{k^2 \mathrm{d}^4 k}{(k^2 + \mu^2)^2} = 2i\pi^2 \mu^2 \log\mu^2 + D_2 + D_3 \mu^2 , \tag{2.16}$$

indicating only the terms $i = j = 0$ in eq. (2.6) explicitly.

The constants $D_{1,2,3}$ depend on the diagram for which the integral is evaluated, but do not depend on μ. Of course, expressions like (2.15) must be handled with great care, but in general they give a very clear idea of where arbitrary numbers enter in the theory. The arbitrariness can only be removed if some additional symmetry property of the system is known, like gauge invariance.

3. MASSLESS YANG-MILLS FIELDS

We now consider the Lagrangian of the massless Yang-Mills theory [10]:

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4} G_{\mu\nu} G_{\mu\nu} , \tag{3.1}$$

$$G_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\,\epsilon_{abc}\, W_\mu^b W_\nu^c , \tag{3.2}$$

which is invariant under the local gauge transformation

$$W_\mu'^a(x) = f_{ab}(x)\, W_\mu^b(x) - \frac{1}{2g}\,\epsilon_{abc}(\partial_\mu f(x) f^{-1}(x))_{cb} . \tag{3.3}$$

If one wants to apply conventional field theory to this model one encounters difficulties [1]. Mandelstam [2] derived Feynman rules for the system using path dependent Green's functions. DeWitt, Faddeev and Popov [3, 4] derived the same rules using a path integral method. We sketch a simple path integral derivation for different gauges in appendix A, and the resulting rules are listed in appendix B:

An auxiliary "ghost particle" appears. In fact it will be seen to cancel the third polarization direction of the W-particles. There is an arbitrariness in gauge, expressed in the parameter λ in the propagator

$$\frac{\delta_{\mu\nu} - \lambda \dfrac{k_\mu k_\nu}{k^2}}{k^2} .$$

Other gauges, like the transversal, can be described in the same way [5].

A path integral derivation of generelized Ward identities is also given in appendix A. A "scalar" W-line
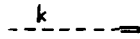
is defined as a W-line with polarization vector $-ik_\mu$:

(3.4)

A "transversal line" has a polarization vector ϵ_μ satisfying

$$k_\mu \epsilon_\mu = 0 ,$$

$$\epsilon_4 = 0 .$$
(3.5)

A generalized Ward identity is then:

= 0.   (3.6)

Amplitudes with "longitudinal W-lines" $(\epsilon_\mu = (-1)^{\delta_{\mu 4}} k_\mu)$ satisfy more complicated Ward identities (cf. sect. 6).

These identities are seen to express the gauge invariance of the theory. For example, the equivalence of the Feynman ($\lambda = 0$) and the Landau gauge ($\lambda = 1$) can be proven using (3.6).

Without much effort one now can verify that the Ward identities are sufficient to prescribe all subtraction constants uniquely, except for the coupling constant. The only needed (and allowed) counterterms are of the following type

$$\delta_{ab}[\delta_{\mu\nu}(C_0 + C_1 k^2) + C_2 k_\mu k_\nu] \ , \tag{3.7a}$$

$$\delta_{ab} C_3 k^2 \ , \tag{3.7b}$$

$$- i g C_4 \epsilon_{abc}[\delta_{\beta\gamma}(q - p)_\alpha + \delta_{\gamma\alpha}(k - q)_\beta + \delta_{\alpha\beta}(p - k)_\gamma] \ , \tag{3.7c}$$

$$- g^2 C_5 [\epsilon_{gac}\epsilon_{gbd}\delta_{\alpha\beta}\delta_{\gamma\delta} + \text{permutations}]$$

$$+ g^2 C_6 [\delta_{ab}\delta_{cd}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta}) + \text{permutations}] \ , \tag{3.7d}$$

$$- i g C_7 q_\alpha \ , \tag{3.7e}$$

(vertices with more φ-lines do not occur because any amplitude must contain as a factor the momenta of the *outgoing* φ-particles (or ingoing φ-antiparticles) as can be seen from the rules (B.1)–(B.6).

The numbers C_1, C_3 and C_4 may be chosen freely, using some convention for the physical amplitude of the W- and φ-fields, and the definition of the physical coupling constant $g^{\text{renormalized}}$. In the Landau gauge moreover, C_2 is immaterial.

According to the Ward identity for the self-energy correction one must have:

$$------\bigcirc------ \ + \ ----\bowtie---- \ = \ 0 \tag{3.8}$$

where the counterterm is indicated explicitly, while

$$----\bowtie---- \ = \delta_{ab}(C_0 k^2 + C_1 k^4 + C_2 k^4) \ .$$

So C_0 is fixed and C_2 is expressed in C_1.

Indeed, an actual calculation of the second-order self energy diagram in the Feynman gauge using the symbolic expressions (2.16) shows:

$$\Pi^{ab}_{\mu\nu} = - \frac{g^2}{(4\pi)^2} \delta_{ab}[\tfrac{10}{3}k^2\delta_{\mu\nu} - \tfrac{10}{3}k_\mu k_\nu] \log k^2 + \delta_{ab}[\delta_{\mu\nu}(C_0 + C_1 k^2) + C_2 k_\mu k_\nu] \ , \tag{3.9}$$

so indeed the Ward identity (3.8) can be satisfied:

$$C_0 = 0, \qquad C_1 + C_2 = 0.$$

The renormalized mass, depending on C_0, turns out to be zero. Note that the coefficients in front of the terms $k^2 \delta_{\mu\nu} \log k^2$ and $k_\mu k_\nu \log k^2$ would not be the same if the φ-particle loop had been left out.

For the four-point function we have,

$$\text{(diagram)} + \text{(diagram)} + \text{(diagram)} + \ \text{-----} \ = 0 \qquad (3.10)$$

while

$$\text{(diagram)} \ \neq\ 0 \quad \text{if} \quad C_5 \quad \text{or} \quad C_6 \neq 0,$$

so C_5 and C_6 are expressed in terms of the other subtraction constants.

Finally, C_7 can be determined by applying the Ward identity (3.8) for the higher order self-energy diagram of the W-particle, using for instance the BPH procedure of renormalization [11], and the above mentioned observation that

$$\text{(diagram)} \ \neq\ 0$$

4. COMBINATORIAL PROOF OF THE WARD IDENTITIES

There is no a priori reason why no conflict situation could emerge if we try to satisfy an infinite number of Ward identities using a finite number of counter terms. This problem must be taken seriously, because the algebraic proof of the Ward identities, which will be given below, involves many shifts of integration variables. A proof of the absence of such a conflict will be given only for one closed loop.

Let us introduce some conventions:

stands for the set of diagrams of a given order in g, and a given number of external transversal W-lines (cf. (3.5)) on mass-shell. There are no longitudinal or scalar external lines. They are denoted explicitly:
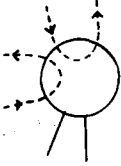
stands for the set of diagrams of a given order in g, and a given number of external transversal W-lines, as above, and in addition a number of external ghost lines and W-lines with arbitrary polarization and momentum, as drawn. The ghost lines are followed inside the graph, which is possible because (B.5) is the only kind of vertex for the ghost particle. The graphs may be disconnected.

The combinatorial proof of the validity of the Ward identities is as follows. From now on we use the Feynman gauge.

Let us perform an infinitesimal gauge transformation in the Lagrangian (3.1):

$$\mathcal{L}_{YM} = -\tfrac{1}{4}G_{\mu\nu}G_{\mu\nu} = \mathcal{L}'_{YM} = -\tfrac{1}{4}G'_{\mu\nu}G'_{\mu\nu}, \tag{4.1}$$

$$W'^{a}_{\mu}(x) = W^{a}_{\mu}(x) + g\,\epsilon_{abc}\Lambda^{b}(x)\,W^{c}_{\mu}(x) - \partial_{\mu}\Lambda^{a}(x), \tag{4.2}$$

Λ is some external source which, according to (4.1), remains uncoupled.

Then we must add to all vertices (B.3) and (B.4) all vertices we get from (B.3) and (B.4) if one of the W-lines ⎯⎯⎯●$_c$ has been substituted by

$$-g\,\epsilon_{abc}\,, \tag{4.3a}$$

$$-\delta_{ac}ik_{\mu}\,. \tag{4.3b}$$

(Note: the double line is not meant to be a propagator; (4.3a) is a part of one vertex). Also from the free part of $\mathcal{L}_{YM}$ we derive an extra vertex term in $\mathcal{L}'_{YM}$, which appears to be

$$-g\,\epsilon_{abc}(\delta_{\mu\nu}p^{2} - p_{\mu}p_{\nu} - \delta_{\mu\nu}q^{2} + q_{\mu}q_{\nu})\,. \tag{4.3c}$$

The ghost particle resulting from the use of a certain gauge condition, is not included in our gauge transformation (4.2). Hence, its vertices and propagators are unchanged.

Now it is easy to verify that up to first order in Λ all extra vertices cancel, which

they should do. In diagrams:

$$= 0 \qquad (4.4a)$$

$$= 0 \qquad (4.4b)$$

$$= 0 \qquad (4.4c)$$

(4.3b) is of the type which occurs in our Ward identities. We now see that it can be replaced by (4.3a) and (4.3c) using eqs. (4.4), except for the connections with the ghost particle. So as

$$(4.5)$$

(Note the explicitly written minus sign for the φ-loop and the combinatory factors, because the blobs are already symmetrized) we have

$$(4.6)$$

eq. (4.6) may be written as

$$\tag{4.7}$$

(Of course, C_1 equals C_2.)

Note that C_3 cancels those diagrams contained in C_1 and C_2 where the double line is attached to a ghost vertex.

The next step is a propagator identity which is related to invariance of the gauge condition under special gauge transformations Λ with $\partial_\mu(\partial_\mu\Lambda^a + g\epsilon_{abc} W_\mu^b\Lambda^c) = 0$:

$$= 0 \tag{4.8a}$$

$$= 0 \tag{4.8b}$$

The P denotes a transversal W-line on mass shell (cf. (3.5)). Note again that the double line is no propagator.

Eq. (4.8a) is the Yang-Mills counterpart of the usual Ward-Takahashi identity (2.12) for bare electron propagators and vertex functions. In the last two terms the dashed line ("Λ-line") has the same vertices and propagators as the ghost particle ("φ-line", compare (B.2) and (B.5)). If some of the lines in (4.8) are parts of a closed loop these identities are true provided one may shift integration variables. This is the reason why subtraction constants must be chosen carefully.

Applying eqs. (4.8) to eq. (4.7) we find

$$\tag{4.9}$$

Eq. (4.9) can now be iterated, but then we must include the possibility that the

Λ-line forms a closed loop and is attached to itself. The result is:

$$(4.10)$$

Using one more identity

$$= 0 \quad (4.11)$$

we have

$$= 0 \qquad\qquad\qquad (4.12)$$

Substituting (4.3b) into (4.3a) one obtains another vertex, for which the following equation holds:

$$= 0 \quad (4.13)$$

Consequently the derivation remains valid even if there are more off-mass shell scalar W-lines:

$$= 0 \qquad\qquad\qquad (4.14a)$$

which is the graphical notation for the formula

$$\frac{\partial}{\partial x_{\mu_1}^1} \cdots \frac{\partial}{\partial x_{\mu_N}^N} \langle \text{out} | T^* (W_{\mu_1}^{a1}(x^1) \dots W_{\mu_N}^{aN}(x^N)) | \text{in} \rangle = 0 , \tag{4.14b}$$

in conventional field theory.

From this algebraic derivation of the Ward identities we draw the following conclusion: if we succeed in regularizing graphs containing one of the auxiliary vertices (4.3a)–(4.3c) in such a way that eqs. (4.8), (4.11) and (4.13) remain valid *also inside closed loops*, then we acquire gauge invariant amplitudes (amplitudes satisfying (4.14)).

5. GAUGE INVARIANT REGULATORS

In this section we construct a set of regulators satisfying all requirements formulated in the previous section, but we confine ourselves to the one closed-loop case. The mere existence of these regulators implies that no conflict situation arises if one uses Ward identities for calculating subtraction constants in the first quantum-mechanical correction, instead of gauge invariant regulators.

The procedure is as follows. Note that the identities (4.8), (4.11) and (4.13) are not only valid in a four-dimensional Minkowsky space, but we may add another dimension. Then the momenta k_μ have five components, and the fields W_μ^a have 15 components. Let for all diagrams with one closed loop the external momenta be in the Minkowsky space, that is, only their first four components differ from zero. Let the momenta inside the closed loop have one more component of fixed length M in a fixed fifth direction. Because of conservation of momentum, M is the same for all propagators of the closed loop. With this interpretation in mind, we may now reformulate the Feynman rules, which now contain an extra parameter M. Furthermore, they depend on which of the propagators belong to the closed loop; those propagators will be denoted by a $*$.

The W- and φ-propagators inside the closed loop are replaced by:

$$\frac{\delta_{ab} \delta_{\mu\nu}}{k^2 + M^2} , \tag{5.1a}$$
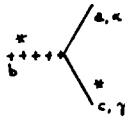
$$\frac{\delta_{ab}}{k^2 + M^2} . \tag{5.1b}$$

The vertices (B.3)–(B.5) remain the same, as well as the propagators (B.1) and (B.2) in the tree parts of a graph. In (5.1a) we let the indices μ, ν run from 1 to 4 as usual. The fifth polarization direction of the W-field is treated as a new particle,

which only occurs inside the closed loop:

$$\frac{\delta_{ab}}{k^2 + M^2}.$$

$$(5.1c)$$

It has the vertices:
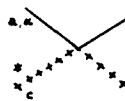
$$Mg\,\epsilon_{abc}\delta_{\alpha\gamma},$$

$$(5.1d)$$

$$-Mg\,\epsilon_{abc},$$

$$(5.1e)$$

(note that the factors $\pm i$ at each end of a crossed line have cancelled), and

$$-ig\,\epsilon_{abc}(q-p)_\alpha,$$

$$(5.1f)$$

$$-g^2(\epsilon_{gac}\epsilon_{gbd} + \epsilon_{gad}\epsilon_{gbc})\delta_{\alpha\beta}.$$

$$(5.1g)$$

Now with vertices (5.1f) and (5.1g) one may have closed loops of crossed lines, but these contributions are gauge invariant themselves, since the vertices (5.1f) and (5.1g) are precisely those of an ordinary isospin one scalar particle. So we may exclude diagrams with closed loops of crossed lines without invalidating the Ward identities. The above vertices with the rule of no closed loop of crossed lines define a set of diagrams which, up to one loop, satisfy the Ward identities. For $M = 0$ we have the diagrams of the massless theory. For M non-zero we have diagrams that may be used as regulator diagrams.

Consider now the sum of diagrams of the massless theory and regulator diagrams. Choosing the appropriate integration variables (remember that each individual contribution may be infinite, and relative shifts of integration variables may give different results) and furthermore regulators with masses M_i and signs e_i, in such a way that

$$\sum_i e_i = 0 , \qquad e_0 = 1 ,$$

$$\sum_i e_i M_i^2 = 0 , \qquad M_0 = 0 ,$$

(5.2)

we obtain a finite result.

One may choose convenient, finite values for

$$\sum_{i \neq 0} e_i \log M_i^2 = -A , \qquad \sum e_i M_i^2 \log M_i^2 = B .$$

(5.3)

In the limit $M_{i \neq 0} \to \infty$ we find the desired gauge invariant amplitudes.

Let us demonstrate this regulator technique for the second order self-energy contributions to the W-propagator:

$$\frac{1}{2} \underbrace{\hspace{1cm}} - \underbrace{\hspace{1cm}} + \underbrace{\hspace{1cm}} + \frac{1}{2} \underbrace{\hspace{1cm}}$$

(5.4)

Using expressions (2.6) we find

$$\Pi_{\mu\nu}^{ab}(k) = \frac{-g^2}{(4\pi)^2} \delta_{ab} \int_0^1 dx \sum_i e_i [\{k^2(5 - 10x(1-x))\delta_{\mu\nu}$$

$$- k_\mu k_\nu (2 + 8x(1-x))\} \log(M_i^2 + x(1-x)k^2)$$

$$- 6M_i^2 \delta_{\mu\nu} \log(M_i^2 + x(1-x)k^2) + 6M_i^2 \delta_{\mu\nu} \log M_i^2] .$$

(5.5)

Indeed, one may convince oneself that this satisfies the Ward identity

$$k_\mu k_\nu \Pi_{\mu\nu}^{ab}(k) = 0 .$$

(5.6)

In the limit $M_{i \neq 0} \to \infty$ we have

$$\Pi_{\mu\nu}^{ab} = - \frac{g^2}{(4\pi)^2} \delta_{ab}(k^2 \delta_{\mu\nu} - k_\mu k_\nu) [\tfrac{10}{3} \log k^2 - \tfrac{10}{3}A - \tfrac{56}{9}] .$$

(5.7)

The number A is the logarithm of a suitably chosen reference mass. It must have the same value for all graphs with one closed loop.

It must be emphasized that even if our regulator method appears very similar to the Pauli-Villars method it is in fact very different. The regulators do not correspond to fields in Lagrangians etc., and the procedure works only for one closed loop. In fact the above is just a convenient way of implementing the scheme pro-

posed in the beginning. Tentative investigation shows that probably a modification of this regulator technique can produce finite gauge invariant amplitudes at higher orders. As yet we shall consider this as a conjecture. It is important to note that this technique of introducing more dimensions only works if the matrix γ^5 and the tensor $\epsilon_{\kappa\lambda\mu\nu}$ do not occur in the Lagrangian.
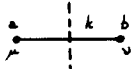
6. UNITARITY

In proving unitarity of the S-matrix one has to deal with on mass-shell amplitudes. We are then confronted with infrared difficulties. Now if we add a very small mass term κ^2 in the propagators, then the on mass-shell amplitudes (in finite order of g) are proportional to some power of $\log \kappa^2$. The Ward identities however, are violated with terms proportional to κ^2, $\kappa^2 \log \kappa^2$, etc. So we can still use these Ward identities keeping $\log \kappa^2$ finite, but ignoring terms proportional to κ^2, $\kappa^2 \log \kappa^2$ etc. For instance, in the regularized expressions in sect. 5 we might put $M_0 = \kappa \neq 0$, but ignore the crossed line with mass κ, because it is coupled with strength κ^2.

We shall not go into the problems of the physical interpretation of these infrared divergencies.
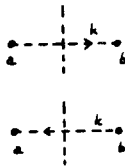
To compute imaginary parts we shall make use of the well-known Cutkosky rules [12] :

$$+ \text{ (graphs with more than two lines cut through)} = 0 , \qquad (6.1)$$

where at the right-hand side of the dashed line the $i\epsilon$ in the propagators is replaced by $-i\epsilon$, and an extra minus sign is introduced for each propagator and each vertex. The blobs are at least of order one in g. Now, if in the blobs of (6.1) all graphs are added, including disconnected ones, such that the total order in g is kept fixed, then equation (6.1) is an identity, whatever the choice of our subtraction coefficients may be, provided that we use the following rules:
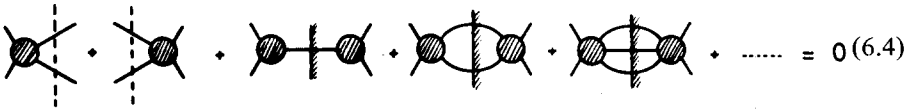
$$2\pi\delta(k^2)\theta(k_0)\delta_{\mu\nu}\delta_{ab} , \qquad (6.2)$$

$$2\pi\delta(k^2)\theta(k_0^2)\delta_{ab} , \qquad (6.3)$$

(a dashed line going through an external particle-line has no special meaning, except that it separates the ingoing lines from the outgoing lines).

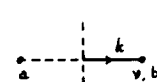Now if we can prove a slightly different equation,
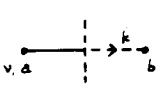

$$\cdots \cdots = 0 \quad (6.4)$$

with

standing for $2\pi\delta(k^2)\theta(k_0)\delta_{ab}\left(\delta_{\mu\nu}-\dfrac{k_\mu k_\nu}{|k|^2}\right)(1-\delta_{\mu4})(1-\delta_{\mu4})$,

$$(6.5)$$

then unitarity has been proven, for the case that bosons with a given isospin have only two helicity states, like the photons. We shall prove eq. (6.4) from eq. (6.1) provided that we only look at the transverse components of the other outgoing lines. Let us first consider the case of only two intermediate particles. Define

$$2\pi\delta(k^2)\theta(k_0)\delta_{ab}\frac{-i\bar{k}_\nu}{2|k|^2} , \qquad \bar{k}_\nu \equiv (-1)^{\delta_{\nu4}}k_\nu$$

$$(6.6)$$

$$2\pi\delta(k^2)\theta(k_0)\delta_{ab}\frac{i\bar{k}_\nu}{2|k|^2} .$$

A useful equation is:

$$\delta_{\mu\nu}=\tfrac{1}{2}\frac{k_\mu\bar{k}_\nu+\bar{k}_\mu k_\nu}{|k|^2}+\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{|k|^2}\right)(1-\delta_{\mu4})(1-\delta_{\nu4}) \qquad \text{if } k^2=0 . \qquad (6.7)$$

Symbolically:

$$\text{(diagram)} = \text{(diagram)} + \text{(diagram)} + \text{(diagram)} \qquad (6.8)$$

Also we have

$$\text{(diagram)} = \text{(diagram)} \qquad (6.9)$$

We shall apply the Ward identities

$$\text{(diagram)} = 0 \ ; \qquad \text{(diagram)} = 0 \ . \qquad (6.10)$$

Moreover, we need a generalization of the Ward identities (4.14) for amplitudes with on mass-shell ghost particles and non-physically polarized W-particles, in particular W-particles with polarization vector e_μ not satisfying $k_\mu e_\mu = 0$. Formula (4.8b) is extended to

$$\text{(diagram)} + \text{(diagram)} + \text{(diagram)} + \text{(diagram)} = 0 \qquad (6.11a)$$

where the arrow in $- - - \!\!\rightarrow \mu$ stands for multiplication with $-ik_\mu$, and the lines with a o are taken on mass-shell ($k^2 = 0$). Note that the last graph in (6.11a) vanishes if multiplied with a transversal polarization vector e_μ. We have also

$$\text{(diagram)} + \text{(diagram)} + \text{(diagram)} = 0 \qquad (6.11b)$$

Applying again the combinatorics of sect. 4 we derive the generalized Ward identity

$$(6.12)$$

(This identity is not altered if other gauge invariant interactions are introduced. The other isospin particles must then be on mass-shell).

Equipped with eqs. (6.8), (6.9), (6.10) and (6.12) we derive

$$(6.13)$$

from which eq. (6.4) follows, as long as we confine ourselves to the contributions with at most two particles in the intermediate states.

In the same way it can be shown for intermediate states with more than two particles that the ghost particles cancel the non-physical polarization directions of the W-bosons. In principle this can be verified by writing down further generalizations of the Ward identity (6.12), but a more straightforward proof of this cancellation goes as follows. We apply induction with respect to the number of particles in the intermediate states.

Suppose we have a diagram
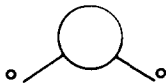
(the external lines being on mass-shell). Let then
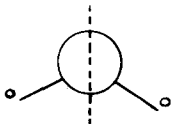
 stand for the sum of all graphs acquired by cutting the former diagram in all possible ways, except that at least one vertex must remain at either side of the dashed line.

Applying again the Cutkosky rule to the left-hand side of (6.12):

$$= 0 \quad (6.14)$$

one derives easily:

$$(6.15)$$

with the external lines on mass shell.

Now careful examination of the underlying propagator identities and combinatorics leads to the observation that eq. (6.15) is also valid if the total number of cut propagators is kept fixed at both sides. So if we introduce the notation

$$(6.16)$$

N denoting the total number of cut propagators, then (6.15) reads:

$$(6.17)$$

for all N. Moreover, one can impose the restriction that the cutting line must pass through both of the explicitly denoted external lines in (6.17), and then we get:

$$(6.18)$$

Now suppose that for a certain value of N

$$(6.19)$$

then we have

$$\text{(6.20)}$$

which completes the proof by induction.

So the S-matrix is unitary in a Hilbert space with only plane wave W-particle states, in which each particle has helicity ± 1. A necessary condition is that subtraction constants are chosen in such a way that all generalized Ward identities are satisfied.

7. CONCLUSION

Massless YM fields can be renormalized. A formal regulator procedure exists, at least for diagrams with one closed loop, but the simplest way to deal with the divergencies is to use the subtracted expressions (2.16) for divergent integrals, calculating subtraction constants by means of the Ward identities. In this article we have not gone into the details of a regulator technique for diagrams with more loops, so as yet a consistency proof of the Ward identity method for removing overlapping divergencies, is lacking.

With this restriction, we have proven that the resulting S-matrix is unitary, if infrared divergencies are dealt with in a proper way. There is only one physical parameter in the theory, which is the coupling constant g. The renormalized mass of the bosons is zero (at least, in perturbation theory).

APPENDIX A

Path integral derivation of Feynman rules for massless Yang-Mills fields

The Feynman path integral expression for the amplitude is

$$\langle \text{out}|\text{in}\rangle = \int \prod_{x,\mu,a} \mathrm{d}W_\mu^a(x) \exp\{i S_{\text{YM}}[W]\}$$

$$\equiv \int \mathcal{D}W \exp i S_{\text{YM}}[W] , \tag{A.1}$$

where a denotes isospin, μ the Lorentz vector component, and $S_{\text{YM}}[W] = \int \mathcal{L}_{\text{YM}}(x)\,\mathrm{d}x$ is the (unrenormalized) Yang-Mills action functional. Now if the asymptotic states are invariant under local gauge transformations Ω, that is

$$\Omega|\text{in}\rangle = |\text{in}\rangle , \qquad \Omega|\text{out}\rangle = |\text{out}\rangle ,$$

then the integrand, as well as the measure $\mathcal{D}W$, are invariant under local gauge transformations.

In order to extract the infinite constant arising from this invariance we alter expression (A.1) by multiplying with a delta-function $\delta(\log\Omega)$ (defined in terms of the same measure $\mathcal{D}W$) where Ω is defined such, that the field

$$W' = \Omega^{-1}(W)$$

satisfies a special gauge condition. We choose the gauge

$$\partial_\mu W_\mu'^a(x) = C^a(x) , \tag{A.2}$$

with $C^a(x)$ a fixed function. Then expression (A.1) becomes

$$\int \mathcal{D}W\,\delta(\log\Omega) \exp i S_{\text{YM}}[W] = \int \mathcal{D}W\,\delta(\partial_\mu W_\mu^a - C^a)$$

$$\times \det\left(\frac{\partial}{\partial\Omega(x')}\,\partial_\mu W_\mu^a(x)\right) \exp i S_{\text{YM}}[W] \tag{A.3}$$

In order to calculate the determinant we only need to know the change of $\partial_\mu W_\mu^a(x)$ under an infinitesimal gauge transformation $\Lambda^b(x)$:

$$\partial_\mu W_\mu^{a'} = \partial_\mu W_\mu^a + \epsilon_{abc}\,\partial_\mu(\Lambda^b W_\mu^c) - g^{-1}\,\partial^2\Lambda^a$$

$$= \partial_\mu W_\mu^a - g^{-1}\,\partial_\mu(D_\mu\Lambda)^a \tag{A.4}$$

(D_μ is the covariant derivative and g is the coupling constant).

So we must calculate the determinant of the operator $g^{-1}\partial_\mu D_\mu$. This we do with the following trick. Note that even for a non-hermitean matrix A_{ij} the identity

$$\frac{1}{\det A} = C \int \prod_i \, \mathrm{d}\,\mathrm{Re}\,z_i \, \mathrm{d}\,\mathrm{Im}\,z_i \, \exp i(z^*, Az) \tag{A.5}$$

holds, where C is a trivial constant. So we write in a symbolic notation eq. (A.3) as

$$\int \mathcal{D}W \, \delta(\partial_\mu W_\mu^a - C^a) \int \mathcal{D}'\varphi \, \exp\{iS_{\mathrm{YM}}[W] + i\int \varphi^*(x)\,\partial_\mu D_\mu \varphi(x)\,\mathrm{d}x\}. \tag{A.6}$$

$\varphi^a(x)$ is a complex scalar particle field. The notation is symbolic because the determinant in eq. (A.3) stands in the numerator and not in the denominator like in eq. (A.5). But this only means that we have to add a factor -1 for each closed loop of φ's, as can easily be established. It is denoted by the prime in $\mathcal{D}'\varphi$.

If C^a is put equal to zero, we get the rules derived by Faddeev and Popov [4]. The transversal propagators

$$\delta_{ab} \frac{\delta_{\mu\nu} - \dfrac{k_\mu k_\nu}{k^2}}{k^2}, \tag{A.7}$$

emerge (Landau gauge)†. We can get rid of the annoying $k_\mu k_\nu$ term by noting that expression (A.6) is completely independent of the choice of $C^a(x)$. So we may integrate over all values of C, together with an arbitrary weight function $\exp iS'[C]$. We then get

$$\int \mathcal{D}W \int \mathcal{D}'\varphi \, \exp\{iS_{\mathrm{YM}}[W] - i\int (\partial_\mu \varphi)^* D_\mu \varphi \,\mathrm{d}x + iS'[\partial_\mu W_\mu]\}. \tag{A.8}$$

$S'[\partial_\mu W_\mu]$ may be chosen such that it cancels the corresponding term in $S_{\mathrm{YM}}[W]$ and we then find the Feynman gauge, with propagators

$$\frac{\delta_{ab}\delta_{\mu\nu}}{k^2}. \tag{A.9}$$

The resulting Feynman rules are listed in appendix B.

Ward identities

We first derive Ward identities in the Landau gauge. Let us treat C^a in expression (A.6) as a source function and make an expansion with respect to it. Even with out-

† The $i\epsilon$ in a propagator is not found by the path integral method. Its sign is dictated by unitarity and is essential for derivation of the Cutkosky rules (sect. 6).

or ingoing particles at plus or minus infinity expression (A.6) is independent of C^a. So all expansion terms with respect to C^a must be zero except the first.

In order to derive the Feynman rules for the expansion terms we must treat the transversal and longitudinal parts of the W-field separately. Integration over the transversal part leads to the Feynman rules (B.1)–(B.6), with $\lambda = 1$, but the fact that $\partial_\mu W_\mu$ now is C and not zero gives us the additional C-lines:

$$\delta_{ab} \frac{-ik_\mu}{k^2} e^{-ikx} , \tag{A.10}$$

where the cross denotes the action of the "source" $C^b(x)$, and the double line simply acts as a normal Yang-Mills boson. (The derivation is done by making $\partial_\mu W_\mu$ variable and adding $-\alpha(\partial_\mu W_\mu^a - C^a)^2$ to the Lagrangian, which gives rise to a delta function for $\alpha \to \infty$.)

We can now formulate our Ward identity in the Landau gauge: *The total contribution of all diagrams with a given (non-zero) number of C-lines off mass-shell, and a given number of in- or outgoing lines on mass-shell, is zero.*

This rule is visualized in the diagram notation (3.6), and corresponds to formula (4.14b).

Eq. (3.6) greatly resembles the corresponding Ward identities in quantum electrodynamics, the only difference being that we have to contract *all* off mass-shell lines with their own momentum (that is, choose a polarization vector proportional to their own momentum). The outgoing lines must be physical, that is, their polarization vector must be orthogonal to their own momentum.

In the Feynman gauge we can do something similar. In expression (A.8) we made the choice

$$S'[C] = \int dx \{ -\tfrac{1}{2} C^2(x) \} .$$
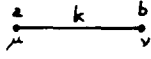
Now we add a source function $J(x)$:

$$S'[C] = \int dx \{ -\tfrac{1}{2} (C(x) - J(x))^2 \} . \tag{A.11}$$

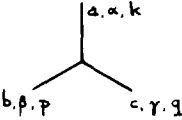Again, the result must be independent of $J(x)$.

The Feynman rules are those of appendix B, with $\lambda = 0$, together with a J-source contribution which is the same as (A.10) except for the (immaterial) factor $1/k^2$. So the Ward identities in this case are again those of eqs. (3.6) and (4.14).
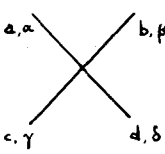
APPENDIX B

Feynman rules for massless Yang-Mills fields

W: $\dfrac{\delta_{ab}}{k^2 - i\epsilon}\left(\delta_{\mu\nu} - \lambda\dfrac{k_\mu k_\nu}{k^2 - i\epsilon}\right)$ $\begin{array}{l}\lambda = 1 \text{ Landau gauge},\\[4pt]\lambda = 0 \text{ Feynman gauge},\end{array}$ (B.1)

ϕ: $\dfrac{\delta_{ab}}{k^2 - i\epsilon}$, (B.2)

  $-ig\,\epsilon_{abc}[\delta_{\beta\gamma}(q-p)_\alpha + \delta_{\gamma\alpha}(k-q)_\beta + \delta_{\alpha\beta}(p-k)_\gamma]$, (B.3)

$-g^2\epsilon_{gac}\epsilon_{gbd}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta})$

$-g^2\epsilon_{gad}\epsilon_{gbc}(\delta_{\alpha\beta}\delta_{\delta\gamma} - \delta_{\alpha\gamma}\delta_{\delta\beta})$

$-g^2\epsilon_{gab}\epsilon_{gcd}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})$, (B.4)

  $-ig\,\epsilon_{abc}q_\alpha$, (B.5)

(at the vertices all momenta are defined to be inwards).

For each closed loop of φ particles: -1 . (B.6)

As usual: a factor $1/(2\pi)^4 i$ for each propagator and $(2\pi)^4 i$ for each vertex.

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