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Quantum Black Holes, Firewalls and the Topology of Space-Time

A pedagogical treatment of a new approach

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Introduction If you want to go into the deep, skip and jump to Part 1, slide 14. But please do read this chapter before criticising the approach.

Without a thorough understanding of the properties of black holes one cannot understand quantum gravity.

This is the 'Lecture Notes' version, and an extension, of my latest papers on black hole microstates. These notes highlight the essentials, such that they should be easier to read and understand. My plan is to keep this updated. If new insights arise, they will be added.

In version v3, a chapter, with figures, on "virtual black holes and space-time foam", was added at the end. In version v4 further remarks were added, as well as several slides to explain better how some apparent contradictions are to be understood, describing in more detail the generic quantum state of a black hole, and how these evolve.

Version v5: More extended introduction, clarifications, responses to criticism, and finally some new features of the current algebra, in particular the relation between mode numbers such as ℓ and m , and the angular momentum.

Version v6: major revisions. Discreteness of the microstate spectrum explained. As I still noted inaccuracies in the earlier texts, I urge readers who hit upon any inconsistencies to mail the author: G.thooft@uu.nl.

Version v7: More texts on motivation, justification and procedure appeared to be needed.

Version v8: Further justification added concerning the black hole formation (and evaporation) process.

Version v9: Further remarks concerning the sign of the commutation rules in region II, slides 51 – 53.

These notes must be seen as a personal view of the author. Most importantly, we try to avoid any wild assumptions, as we usually see in the literature. Our claim is that everything here is essentially derived from basic laws that are well-established. Not even String Theory is used (this might change in the future).

These notes are not intended for use in talks (too much information on single “slides”) with apologies for not always following the same notation as used elsewhere.



See: G. 't Hooft, *Found. Phys.*, 49(9), 1185-1198 (2016), DOI 10.1007/s10701-016-0014-y;
47 (12) 1503-1542 (2017) DOI 10.1007/s10701-017-0122-3;

arxiv: 1601.03447v4[gr-qc],

arxiv: 1605.05119 [gr-qc],

arxiv: 1612.08640 [gr-qc],

arxiv: 1804.05744 [gr-qc],

arxiv: 1809.05367 [gr-qc],

arxiv: 1902.10469 [gr-qc].

+ references there.

The approach explained here is not intended to *replace* other approaches, such as pure string theory descriptions or “fuzzballs”. Rather, our approach is complementary. String descriptions, fuzzballs and other attempts, all have in common that unconventional degrees of freedom are called upon because it is thought that new ‘Planck scale physics’ should be necessary to address the information paradox.

Alternatively, however, one could ask how *conventional physics* could be used to address this purported paradox. Why not consider the gravitational interactions between in- and out-going material? We find that this makes a **big** difference.

Hawking radiation consists predominantly of particles whose energies are far below the Planck value, and this is why one might attempt to formulate the effective laws of a black hole exclusively in terms of low-energy physics, of which the rules are well-documented: the Standard Model, augmented with perturbative quantum gravity.

The Hilbert space of black hole states that we consider primarily consists of a fixed background metric with excitations in the form of low-energy particles including gravitons.

But that is not the entire story: due to gravitational interactions, Hilbert space is also affected by 'cut-and-paste' procedures in space-time. We call this the 'firewall transformation' and it needs further justification. This is arguably the biggest departure from conventional physics, but I find it particularly elegant and beautiful. When the dust settles, the *firewalls* have disappeared.

One might object that Planck scale phenomena will probably enter through the back door. To our astonishment however, we found this not to be the case. The physical degrees of freedom that are generated by the gravitational back reaction obey *linear equations that diagonalise*, when described in terms of the proper variables: the spherical harmonics.

If we would try to take into account matter that formed the black hole in the distant past, or matter in terms of which a black hole evaporates entirely in the distant future, one finds that these represent degrees of freedom that, at intermediate times, obtain energies far beyond the Planck scale, and consequently, for describing *pure* quantum states, we should *not* include these degrees of freedom, as will be explained further.

At first sight, our Hilbert space consisting only of low-energy particles ('soft particles') on a smooth background can only be maintained for a short amount of time as seen by the distant observer. But we made an amazing discovery: we can map states at (somewhat) later times 1-1 on states at earlier times, and by repeating this, we can reach all time epochs; a state in our Hilbert space *stays* in our Hilbert space.

This is what we called 'cut-and-paste'.

At each *given* time, our procedure starts with only a narrow time segment. In that time segment, we must use the metric of a *permanent* black hole. The Penrose diagram for this *permanent* black hole has 4 regions, *I*, *II*, *III*, and *IV*. One then hits upon one novelty in our work: after the cut-and-paste, regions *II* and *IV* are identified as the *antipodes* of regions *I* and *III* (obtained by mirroring the three coordinates x , y and z along the horizon).

The above *antipodal identification* is mandatory. Without it, our evolution law would not be unitary.

It replaces the ER=EPR assumption.

As soon as we adopt this constraint, as a new boundary condition for the eternal Penrose diagram, we find a set of eigen states of our evolution law. No further assumptions are now needed concerning the Planck scale.

The Planck scale does enter when we go to the high ℓ spherical modes, but, as the equations are linear and decouple (in the approximations used – and justified), we may include all ℓ up to a freely adjustable cut-off. The best way to choose this cut-off is by allowing exactly the number of microstates as is needed to reproduce Hawking's value.

Apart from the above, just known principles, such as Quantum Mechanics and General Relativity, were used.

Note, that QM and GR are also assumptions, but they are much more compelling. Dropping or loosening these assumptions is much more difficult than keeping them as long as we can. No theory is sacred, and some sense of taste for what to keep and what not, is a useful attire for a theoretical physicist.

Topics such as microstates can be understood very well in our theory.
(As for the exact number of orthonormal microstates, our theory does allow us to count them, see arXiv:1809.05367)

In this work, we focus on the Schwarzschild black hole, but generalisation to Reissner-Nordström, Kerr, or Kerr-Newman seems to be straightforward. The BPS limit towards extreme black holes will require more care.

The picture of quantum black holes given in several publications suggests them to be chaotic systems. In our work, chaos may only occur at large time scales. We are more interested in short time scales, where chaos does not yet enter.

Note, that the 'fuzzy' or 'stringy' shape of the horizon in the 'fuzzball' theory assumes that the massive string modes are excited at the horizon. However, the massive string modes are much more energetic than the modes we need for our procedure. Thus, as long as ℓ values are not too high, the states in Hilbert space that we consider, should not be affected by such modes.

Quantum gravity in the bulk region, up to very near the event horizons, can be handled by standard perturbation expansion.

In Planck units, all out-going particles will have energies comparable to the *inverse* of the black hole mass. And of course, we can keep all in-going particles in that domain as well. Then all perturbations on the black hole's space-time metric will be small. To accommodate for these perturbations we include gravitons among the low energy particles.

Thus we shall obtain an S matrix that will be shown to be unitary *in the Hilbert space spanned by particles with energies (far) below the Planck mass* (we may choose the limit of ℓ so low that considerably fewer micro-states are involved, but if we do that, an other problem arises, see beginning of slide 12).

Only at the horizon itself, one has to take into account the *gravitational back reaction*. Surprisingly, this entirely non-perturbative feature can be handled, since it happens to be described by linear equations.

Indeed, we emphasise that one can use ordinary coordinates in the bulk region, while at the (past as well as future) event horizons, the gravitational back reaction *saves the theory from breaking down*.

What used to be thought of as chaos is now brought to order. Our system is comparable to a large number of quantum particles in a multi-dimensional rectangular box, with non-rational ratios in the lengths of the edges and the momenta of the particles. If viewed as a gas, it looks chaotic. But it is not.

In such a box, the particles do not interact, so all of their motions, in all directions, are decoupled. Every single particle, in every of the principal directions in the box, is described by a simple, one dimensional (quantum) equation of motion, which is trivial to solve for an undergraduate student. The black hole is just like this. Particles in the bulk interact, but only weakly, so, in our first approximations, we may neglect their interactions

Due to the fact that, on the horizons, we can expand the distribution of in- and out-going matter in *spherical harmonics*, we obtain equations that show that, on the horizons, all these different spherical harmonic modes almost completely decouple. Their equations factorise, just as the particles in our toy example, or the quantum modes in the hydrogen atom.

The gravitational back reaction also factorises. The great advantage of using this feature is that now we can easily solve the mathematical equations and see what happens. Does the antipodal identification make sense at all? Is it needed for restoring unitarity?

At first sight, it seems that what we do is plainly wrong, and it was criticised as such in my email exchanges with colleagues. But the equations are so simple that one can investigate all alternatives.

It is essential to have Cauchy surfaces, and examine how the data on these surfaces evolve with time. What is the time coordinate? How can we ensure that the Cauchy data evolve through unitary equations? The Cauchy data should not get lost between the crevices of the horizons, which can easily happen while you think you are doing things correctly.

An important step has not yet been discussed. We explain how to obtain a unitary S matrix, expressed explicitly in terms of the orthonormal basis of a specially chosen Hilbert space. But this raises a question: *How should we map this orthonormal basis, defined in terms of the momentum distribution of in- and out-going matter, onto the orthonormal basis generated by the Fock space of the Standard Model (or whatever replaces that at the scale where we consider the black hole)?*

This mapping must be 1-1, but it is non-linear.

This problem seems to be very difficult. However, in ancient publications¹, the author demonstrated the close resemblance of our theory to a (Wick-rotated) string theory: the black hole horizon is mathematically closely related to a string theory world sheet. How do strings interact with particles coming in and out, at relatively low energies? Surprisingly, this difficulty is very similar to our difficulty with the 1-1 mapping sought for above. *And string theorists solved it!* The solution is that the state of N external particles is generated by putting N vertex insertions on the world sheet. Replace these vertex insertions by N Dirac delta peaks of in-going (or out-going) momentum, and then integrate over the positions of these vertices over the horizon.

¹G. 't Hooft, The scattering matrix approach for the quantum black hole: an overview, J. Mod. Phys. A11 (1996) pp. 4623-4688. gr-qc/9607022, and references therein.

Thus, we should copy the string theory solution into our horizon theory. It should be straightforward, but it has not been done explicitly. This time, we insist that the mapping should be a unitary one.

But yes, we have an important problem: there are practically no experiments.

We conclude that we must be extremely careful and use devices such as spherical wave expansions or whatever else we can put our hands on, to get a clearer picture and to simplify our subject.

To my opinion, such advice should be followed meticulously.

See here how far I got. This author however, is still working mostly alone, and yes, I can make mistakes. If you, reader, think I made mistake(s) please do not hesitate to contact g.thoof@uu.nl

PART 1, Theoretical considerations and derivations

The tortoise coordinates

Our prototype is the Schwarzschild black hole. No serious complications are expected when generalised to Kerr-Newman or such. The *extreme* (BCS) black hole would not serve our purposes, because it is a limiting case, and its horizon is fundamentally different from the more generic black holes.

Consider the Schwarzschild metric:

$$ds^2 = \frac{1}{1 - \frac{2GM}{r}} dr^2 - \left(1 - \frac{2GM}{r}\right) dt^2 + r^2 d\Omega^2 ; \quad \begin{cases} \Omega & \equiv (\theta, \varphi) , \\ d\Omega & \equiv (d\theta, \sin\theta d\varphi) . \end{cases}$$

Go to Kruskal-Szekeres coordinates x, y , defined by

$$\begin{aligned} xy &= \left(\frac{r}{2GM} - 1\right) e^{r/2GM} ; \\ y/x &= e^{t/2GM} . \\ ds^2 &= \frac{32(GM)^3}{r} e^{-r/2GM} dx dy + r^2 d\Omega^2 . \end{aligned}$$

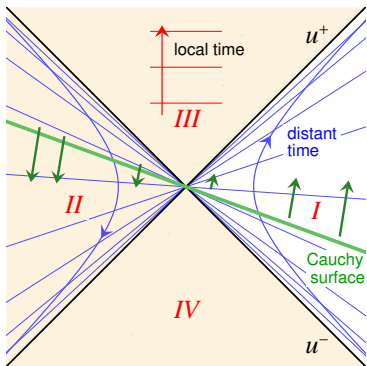
At $r = 2GM$, we have $x = 0$: past event horizon, and
 $y = 0$: future event horizon.

We concentrate on the region close to the horizon: $r \approx 2GM$. There:

$$x \rightarrow \frac{\sqrt{e/2}}{2GM} u^+ ; \quad y \rightarrow \frac{\sqrt{e/2}}{2GM} u^- ; \quad 2GM \equiv R$$

$$ds^2 \rightarrow 2du^+ du^- + R^2 d\Omega^2 .$$

$$\text{time } t/4GM = \tau$$



$$u^-(\tau) = u^-(0)e^\tau$$

$$u^+(\tau) = u^+(0)e^{-\tau}$$

As time goes forwards, u^+ approaches the horizon asymptotically;

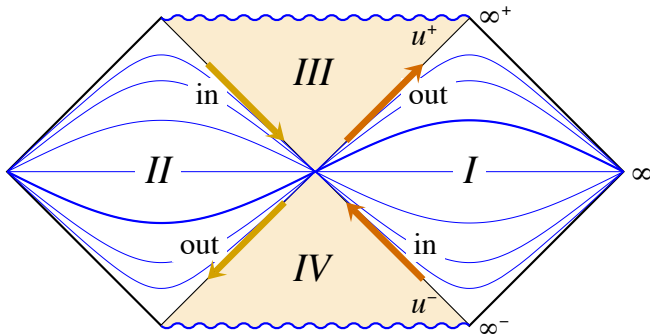
as time goes backwards, u^- approaches the past horizon asymptotically (tortoises).

Particles going in generate wave functions on u^+ , particles going out start as wave functions on u^- .

If we redefine $u^+ \rightarrow f(u^+)$, $u^- \rightarrow g(u^-)$, then the metric keeps the form $ds^2 = 2A(u)du^+du^- + r^2(u)d\Omega^2$. \rightarrow *Map u^\pm on compact domains:*

The Penrose diagram.

For pure Schwarzschild (without matter either responsible for the formation of a black hole, or representing its final decay):



Now, we begin to deviate from standard practice:

This is the “eternal black hole” (it never formed, it will never decay, since, on the average, as many particles are going in, as there are going out). We keep it. Why?

We put the “microstates” on a Cauchy surface. As yet, we are only interested in how they evolve for a short amount of time. For the quantum properties of the BH, the distant past, and the distant future, should be irrelevant.

An elegant way to justify this: in case the implosion took place infinitely long ago, the out-going Hawking particles will be only in one state, the Hartle-Hawking state (see Slide # 57). When we observe the out-going Hawking particles, we project this state along a much larger class of basis states. Extrapolating these states back to the past, we will see many more recent in-going particles, including more recent implosions. We do allow all these states, but we identify them by exclusively counting the recent in- and out-going particles only,

which will be further justified a posteriori.

This picture, however, is yet to be considered as an intermediate state of our discussion. We use it as a starting point to describe small variations in the black hole state during small periods in time. Later, we will address the black hole's long-term time dependence more accurately, and find the above formalism to be perfectly fitting for starting our procedure.

Hard and soft particles

First, consider the particles near the horizon(s), having momenta (p^+, p^-, \tilde{p}) . The mass shell condition is $2p^+p^- + \tilde{p}^2 + \mu^2 = 0$.

Here, \tilde{p} is the transverse part of the momentum, $\mu = \text{mass}$;

$|\tilde{p}| \approx L/R$ and μ are basically constant.

But $p^-(\tau) = p^-(0)e^\tau$, $p^+(\tau) = p^+(0)e^{-\tau}$ (See slide 16)

Define **soft** particles: $|\tilde{p}|, \mu \ll M_{\text{Planck}}$ Negligible effect on space-time curvature.

Define **hard** particles as particles that do cause appreciable space-time curvature.

As $\tau \rightarrow \infty$, $p^+ \rightarrow 0$, $p^- \rightarrow \infty$: all in-particles will become **hard**;

As $\tau \rightarrow -\infty$, $p^- \rightarrow 0$, $p^+ \rightarrow \infty$: all out-particles originally were **hard**.

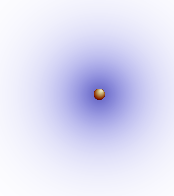
$|\tilde{p}|$ and μ stay small.

To understand what happens with the evolution at longer time intervals, we have to understand what the hard in- and out- particles do.

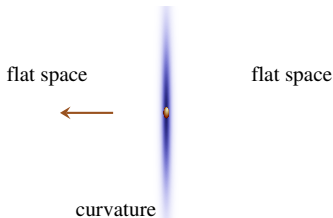
Consider an in-particle that is soft at $\tau \approx 1$. Now consider the same particle in a coordinate system centering at a later time $\tau \gg 1$. The in-particle then has become hard, since its momentum increases exponentially with time. Its interactions with other in-particles are negligible (they basically move in parallel orbits), but it does interact with the out-particles. The interaction through QFT forces stay weak, but the *gravitational forces* make that (early) in-particles interact strongly with (late) out-particles.

The *gravitational effect* of a fast, massless particle is easy to understand:

Schwarzschild metric of a particle with tiny rest mass $m \ll M_{\text{Planck}}$:



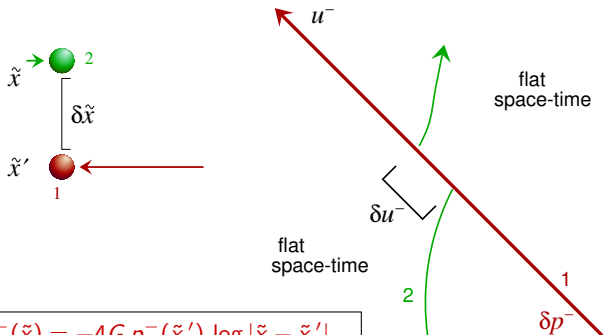
And now apply a strong Lorentz boost, so that $E/c^2 \gg M_{\text{Planck}}$:



This we use to compute the gravitational force between hard and soft particles;

The gravitational back reaction:

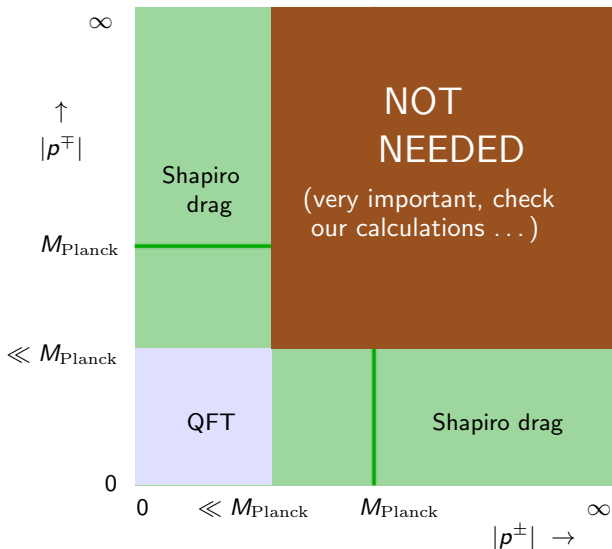
Calculate the *Shapiro time delay* caused by the grav. field of a fast moving particle:
 simply Lorentz boost the field of a particle at rest:



$$\delta u^-(\tilde{x}) = -4G p^-(\tilde{x}') \log |\tilde{x} - \tilde{x}'| .$$

P.C. Aichelburg and R.U. Sexl, *J. Gen. Rel. Grav.* **2** (1971) 303,
 W.B. Bonnor, *Commun. Math. Phys.* **13** (1969) 163,
 T. Dray and G. 't Hooft, *Nucl. Phys.* **B253** (1985) 173.

What we need to know is the interactions between in- and out-particles.



The fact that the mutual interactions between hard particles, at the Planck mass or beyond, will not be needed, is a very important aspect of this work. As you will see.

We start with only soft particles on a Cauchy surface of the Penrose diagram. These will define all quantum microstates of the black hole at a given time.

Now, the question is how do these evolve with time.

The soft particles won't stay soft; their longitudinal momenta will quickly explode. To see what happens, calculate δu^- (the Shapiro shift).

For that, we use Slide # 22, adapted to the Schwarzschild metric. This means that we replace the transverse coordinates \tilde{x} by the solid angle $\Omega = (\theta, \varphi)$.

An in-particle with momentum p^- at solid angle $\Omega' = (\theta', \varphi')$
causes a shift δu^- at solid angle $\Omega = (\theta, \varphi)$:

$$\delta u^-(\Omega) = 8\pi G f(\Omega, \Omega') p^- ; \quad (1 - \Delta_\Omega) f(\Omega, \Omega') = \delta^2(\Omega, \Omega') .$$

If there are many in-particles:

$$p^-(\Omega) \equiv \sum_i p_i^- \delta^2(\Omega, \Omega_i)$$
$$\delta u^-(\Omega) \equiv 8\pi G \int d^2\Omega' f(\Omega, \Omega') p^-(\Omega') .$$

Our “reference” black hole will have a few soft particles seen by a *local* observer. This quantum state will be close to the Hartle Hawking state (see slide 57).

The *distant* observer will see unending streams of in- and out-particles with given positions u^+ or u^- . Suppose now that $p^-(\Omega)$ represents *all* in-particles needed to describe any black hole in a given quantum state.

This means that we replace δu^- by u^- itself.

Later, we will see how imploding matter may be described starting with momentum distributions p^- at the distant past.

Then the out-particles will be at positions $u^-(\Omega)$ given by

$$u_{\text{out}}^-(\Omega) = 8\pi G \int d^2\Omega' f(\Omega, \Omega') p_{\text{in}}^-(\Omega') .$$

Notice that *very early* in-going particles have huge values for p^- . Therefore they generate out-particles with huge values of u^- – these must have separated from the black hole long ago! We return to this subject later.

Spherical wave expansion:

$$u^\pm(\Omega) \equiv \sum_{\ell, m} u_{\ell m}^\pm Y_{\ell m}(\Omega) , \quad p^\pm(\Omega) \equiv \sum_{\ell, m} p_{\ell m}^\pm Y_{\ell m}(\Omega) ;$$

$$[u^\pm(\Omega), p^\mp(\Omega')] = i\delta^2(\Omega, \Omega') , \quad [u_{\ell m}^\pm, p_{\ell' m'}^\mp] = i\delta_{\ell\ell'}\delta_{mm'} ; \quad (1)$$

$$u_{\text{out}}^- = \frac{8\pi G}{\ell^2 + \ell + 1} p_{\text{in}}^- , \quad u_{\text{in}}^+ = -\frac{8\pi G}{\ell^2 + \ell + 1} p_{\text{out}}^+ ,$$

$p_{\ell m}^\pm$ = total momentum in of in^{out} -particles in (ℓ, m) -wave ,

$u_{\ell m}^\pm$ = (ℓ, m) -component of c.m. position of in^{out} -particles .

Because we have linear equations, all different ℓ, m waves decouple, and for one (ℓ, m) -mode we have just the variables u^\pm and p^\pm . They represent only one independent coordinate u^+ , with $p^- = -i\partial/\partial u^+$.

The dynamics completely factorises in (ℓ, m) spherical harmonics.

This complete decoupling is due to our approximations, which are very good as long as $\ell \ll M_{\text{BH}}$... in Planck units.

This remains true in the harmonics of a Kerr black hole, but that generalisation is not considered here.

3 more steps to be taken:

1. Starting with the QFT states of the particles entering the future event horizon, we must calculate their $p^-(\theta, \varphi)$ distribution there. The high momentum cut-offs must be chosen such that this mapping is unitary (or: reversible).

This is difficult.

Comparable to vertex insertions as in string theory.

2. $p^-(\theta, \varphi)$ on the future event horizon generates $u^-(\theta, \varphi)$ on the past event horizon. But its support is $[-\infty, +\infty]$, so it is spread over both regions *I* and *II*. What is the physical interpretation of region *II* ?

Postulate that region *II* refers to the same black hole as region *I*, but *not at the same solid angle* $\Omega = (\theta, \varphi)$. Only one possibility:

The antipodal identification:

$$\begin{array}{l} I \leftrightarrow II \\ \Omega \leftrightarrow \tilde{\Omega} = (\pi - \theta, \varphi + \pi) \end{array}$$

3. The “firewalls”. Soft particles become hard particles. Must be ‘removed’.

Obvious suggestion: all information carried by the in- particles is now present in the out-particles. The in-particles are redundant (“quantum clones”). Leave the hard ones out. Hilbert space is completely specified by the coordinates $u^-(\Omega)$ of the out-particles, as soon as $|u^-| > L_{\text{Planck}}$, these out-particles are then soft.

This is the “firewall transformation”; it removes all hard particles in the longitudinal direction, and thus removes the firewalls.

The firewall transformation is what we call a cut-and-paste procedure: spacetime is cut along the horizon, and pasted back with the shift δu^- . This removes the in-particle, but not the information it carries; this information is imparted to all out-particles, which are shifted by the amount δu^- .

We can subsequently attribute the positions of all out-particles to cut-and-paste procedures at different moments in time. The out-particles carry the *footprints* of the in-particles.

The antipodal identification requires the constraint that the local laws of nature are invariant across the seam. This operation has $C = P = +1$, $T = CPT = -1$.

As is explained in slide # 41, this requires that the local QFT is invariant under CPT transformations.

The basic, explicit, calculation

The algebra (1), slide 26, generates the scattering matrix, by giving us the *boundary condition* that replaces $|\text{in}\rangle$ -states by $|\text{out}\rangle$ -states. This boundary condition **replaces the old *brick wall model*** and, in the **spherical harmonics expansion**, it is embarrassingly easy to derive and use.

All of this is NOT a model, or a theory, or an assumption, but a calculation . . .

Apart from the most basic assumption of unitary evolution, which forces us to fold the Penrose diagram along the antipodes, this is nothing more than applying GR and quantum mechanics !

Note that the antipodal identification comes together with the cut-and-paste procedure.

At some points in the following derivations, it may seem that choices were made as to how to continue, such as the eternal BH Penrose diagram and the antipodal identification, and some of my choices are being criticised as they may seem illogical.

I emphasise however that they will all be vindicated by the final result, a unitary evolution law. There is no other way to obtain that. The internal logic will become clear. None of the other approaches in the literature that I have seen produce such explicit results.

At every (ℓ, φ) , we have just one variable p^- , proportional to one u^- , with the Fourier transform p^+ , proportional to u^+ .

So we have 1 dimensional quantum mechanics !

Now let us deduce what happens with the Fourier transformation when we use tortoise coordinates.

Let there be two operators, u and p , obeying the commutator equation

$$[u, p] = i, \quad \text{so that} \quad \langle u|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipu}.$$

and a wave function $|\psi\rangle$, defined by $\psi(u) \equiv \langle u|\psi\rangle$. Its Fourier transform is

$$\hat{\psi}(p) \equiv \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du e^{-ipu} \psi(u).$$

Now introduce **tortoise coordinates**, and split both u and p in a **positive part** and a **negative part**:

$$u \equiv \sigma_u e^{\varrho_u}, \quad p = \sigma_p e^{\varrho_p}; \quad \sigma_u = \pm 1, \quad \sigma_p = \pm 1, \quad \text{and}$$

$$\tilde{\psi}_{\sigma_u}(\varrho_u) \equiv e^{\frac{1}{2}\varrho_u} \psi(\sigma_u e^{\varrho_u}), \quad \tilde{\psi}_{\sigma_p}(\varrho_p) \equiv e^{\frac{1}{2}\varrho_p} \hat{\psi}(\sigma_p e^{\varrho_p});$$

normalisation requires:

$$|\psi|^2 = \sum_{\sigma_u=\pm} \int_{-\infty}^{\infty} d\varrho_u |\tilde{\psi}_{\sigma_u}(\varrho_u)|^2 = \sum_{\sigma_p=\pm} \int_{-\infty}^{\infty} d\varrho_p |\tilde{\psi}_{\sigma_p}(\varrho_p)|^2. \tag{2}$$

What is the Fourier transform in these tortoise coordinates?

$$\tilde{\psi}_{\sigma_p}(\varrho_p) = \sum_{\sigma_u=\pm 1} \int_{-\infty}^{\infty} d\varrho_u K_{\sigma_u\sigma_p}(\varrho_u + \varrho_p) \tilde{\psi}_{\sigma_u}(\varrho_u) ,$$

with $K_{\sigma}(\varrho) \equiv \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\varrho} e^{-i\sigma} e^{\varrho} .$

Notice the symmetry under $\varrho_u \rightarrow \varrho_u + \lambda$, $\varrho_p \rightarrow \varrho_p - \lambda$, which is simply the symmetry $u \rightarrow u e^{\lambda}$, $p \rightarrow p e^{-\lambda}$, a property of the Fourier transform, a consequence of *time translation invariance w.r.t. the external observer*, and an invariance of our algebra.

We now use this symmetry to write plane waves:

$$\tilde{\psi}_{\sigma_u}(\varrho_u) \equiv \check{\psi}_{\sigma_u}(\kappa) e^{-i\kappa\varrho_u} \quad \text{and} \quad \tilde{\psi}_{\sigma_p}(\varrho_p) \equiv \check{\psi}_{\sigma_p}(\kappa) e^{i\kappa\varrho_p} \quad \text{with}$$

$$\check{\psi}_{\sigma_p}(\kappa) = \sum_{\sigma_p=\pm 1} F_{\sigma_u\sigma_p}(\kappa) \check{\psi}_{\sigma_u}(\kappa) ; \quad F_{\sigma}(\kappa) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K_{\sigma}(\varrho) e^{-i\kappa\varrho} d\varrho .$$

Thus, we see left-going waves produce right-going waves.

The Fourier transform in x , p space is non-local:

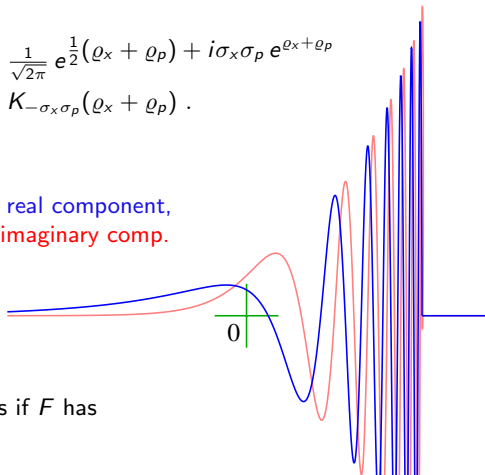
$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}$$

But if we write $x = \sigma_x e^{\varrho_x}$ and $p = \sigma_p e^{\varrho_p}$, where σ_x and σ_p are signs \pm , then the relation becomes:

$$\begin{aligned}\langle \varrho_x, \sigma_x | \varrho_p, \sigma_p \rangle &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}(\varrho_x + \varrho_p) + i\sigma_x\sigma_p e^{\varrho_x + \varrho_p}} \\ &= K_{-\sigma_x\sigma_p}(\varrho_x + \varrho_p) .\end{aligned}$$

Blue = real component,
Red = imaginary comp.

$K_+(x)$:



In practice it will appear as if F has a finite support.

Look at how our soft particle wave functions evolve with time τ , slide # 16 or 19.

Their Hamiltonian is the dilaton operator. Let κ be the energy:

$$H = -\frac{1}{2}(u^+ p^- + p^- u^+) = \frac{1}{2}(u^- p^+ + p^+ u^-) =$$
$$i \frac{\partial}{\partial \varrho_{u^+}} = -i \frac{\partial}{\partial \varrho_{u^-}} = -i \frac{\partial}{\partial \varrho_{p^-}} = i \frac{\partial}{\partial \varrho_{p^+}} = \kappa .$$

The energy eigen states are $C (p^-)^{i\kappa} = C e^{i\kappa \varrho_{p^-}}$,

The Fourier operator on these states (σ is sign difference) is:

$$F_\sigma(\kappa) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dy}{y} y^{\frac{1}{2}-i\kappa} e^{-i\sigma y} = \frac{1}{\sqrt{2\pi}} \Gamma\left(\frac{1}{2} - i\kappa\right) e^{-\frac{i\sigma\pi}{4} - \frac{\pi}{2}\kappa\sigma} .$$

Matrix $\begin{pmatrix} F_+ & F_- \\ F_- & F_+ \end{pmatrix}$ is unitary: $F_+ F_-^* = -F_- F_+^*$ and $|F_+|^2 + |F_-|^2 = 1$.

The scattering matrix

Add the scale factor $\frac{8\pi G}{\ell^2 + \ell + 1}$, to get, if $u^\pm = \sigma_\pm e^{\rho^\pm}$,

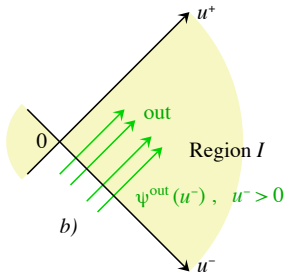
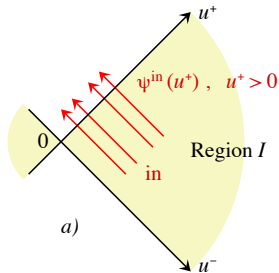
$$\psi_{\sigma_+}^{\text{in}} e^{-i\kappa\rho^+} \rightarrow \psi_{\sigma_-}^{\text{out}} e^{i\kappa\rho^-},$$

$$\begin{pmatrix} \psi_+^{\text{out}} \\ \psi_-^{\text{out}} \end{pmatrix} = \begin{pmatrix} F_+(\kappa) & F_-(\kappa) \\ F_-(\kappa) & F_+(\kappa) \end{pmatrix} e^{-i\kappa \log(8\pi G/(\ell^2 + \ell + 1))} \begin{pmatrix} \psi_+^{\text{in}} \\ \psi_-^{\text{in}} \end{pmatrix}. \quad (3)$$

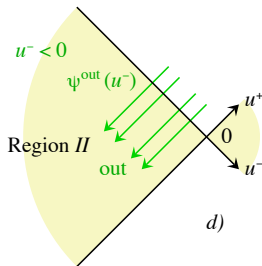
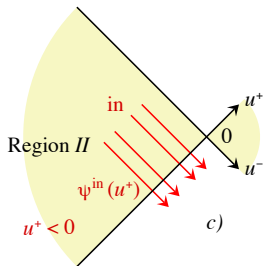
These equations generate the contributions to the scattering matrix from all (ℓ, m) sectors of the system, where $|m| \leq \ell$. At every (ℓ, m) , we have a contribution to the position operators $u^\pm(\theta, \varphi)$ and momentum operators $p^\pm(\theta, \varphi)$ proportional to the partial wave function $Y_{\ell m}(\theta, \varphi)$. The signs of $u^\pm(\theta, \varphi)$ tell us whether we are in region *I* or region *II*. The signs of $p^\pm(\theta, \varphi)$ tell us whether we added or subtracted a particle from region *I* or region *II*.

In Slides # 46 – 47, we derive that, considering all (ℓ, m) values with $\ell < \ell_0 = \mathcal{O}(M_{\text{BH}}/M_{\text{Planck}})$ (the angular momentum limit), gives us $\approx e^{c^{\text{nst}} \ell_0^2 \log \ell_0^2}$ microstates, which is the right order of magnitude. The microstates will turn out to be discrete, but note that this black hole is not exactly thermal, due to antipodal entanglement.

- a) Wave functions $\psi(u^+)$ of the in-particles live in region I, therefore $u^+ > 0$.
 b) Out-particles in region I have $\psi(u^-)$ with $u^- > 0$.



The physical picture



- c, d) In region II, the in-particles have $u^+ < 0$ and the out-particles $u^- < 0$.

Note that the in-particles will never get the opportunity to become truly hard particles.

Eq. (3) is to be seen as a “soft wall”-boundary condition near the origin of the Penrose diagram. Wave functions going in are reflected as wave functions going out. These again emerge as soft particles.

Thus, there is no firewall, ever.

In the previous slide, the total of the in-particles in regions *I* and *II* are transformed (basically just a Fourier transform) into out-particles in the same two regions.

Note that the regions *III* and *IV* in the Penrose diagram (see slide 17) never play much of a role, even if an observer falling in region *III* would want to assure us that (s)he is still alive.

These regions are best to be seen as lying somewhere on the time-line *where time t is somewhere beyond infinity* (thus a mere repetition of the same degrees of freedom at finite time), which is why we have no 'over-counting of quantum copies'.

The catch?

Our “boundary condition at the origin” is in terms of momentum distributions $p^\pm(\theta, \varphi)$ and center-of-mass positions $u^\pm(\theta, \varphi)$ only.

But we would need the quantum wave functions as elements of Fock space (which would be specified by positions or momenta, but also by other quantum numbers!)

Such mapping should be unitary in Hilbert space. We are not certain that this can be done, but it is natural to look at how it is done in string theory. Our momentum distributions are like vertex insertions, although they are on the horizon instead of the string world sheet.

Presumably we are describing black holes with variable sizes. There may well be a cut-off at some large ℓ values depending on M . It can be argued that the microstates do form a *discrete set*, even though the wave functions seem to be continuous. This we explain In Slides # 46 – 47.

The antipodal identification

Regions I and II of the Penrose diagram are exact copies of one another. Often, it was thought that region II describes something like the 'inside' of a black hole. That cannot be right, since region II , like region I , has asymptotic regions. Hawking suggested that region II might be some other black hole, in an other universe, or far away in our universe. However, our 2×2 scattering matrix implies that the two regions are in contact with each other quantum mechanically. In ordinary branches of physics, such long-distance communication never takes place, and I don't think theories with such features make any sense.

It is far more natural to assume that region II describes the same black hole as region I . It must then represent some other part of the same black hole. Which other part? The local geometry stays the same, while the square of this $O(3)$ operator must be the identity.

There is exactly one possibility: This is the $O(3)$ operator $-\mathbb{I}$, which is:
the antipodal mapping.

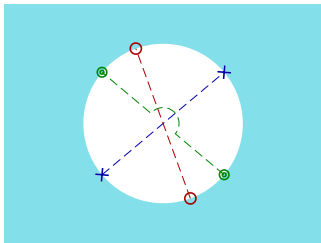
Thus the antipodal mapping seems to be a parity inversion, but when we combine it with an inversion of the coordinate dr , the parity is positive. Keeping the C charge of particles also unchanged we see that only time T is inverted. QFT admits no forces that change when only T or CPT is switched while C and P themselves stay the same. This is CPT invariance. Antipodal *identification* only holds for the central point (origin) of the Penrose diagram. Regions I and II are different regions of the universe. But relating region II to region I by demanding that the angular coordinates are antipodes, means that now the mapping from Schwarzschild coordinates to Kruskal Szekeres coordinates is one-to-one. This now turns out to be an essential property of our coordinate transformations. Thus, we arrive at a new restriction for all general coordinate transformations:

In applying general coordinate transformations for quantised fields on a curved space-time background, to use them as a valid model for a physical quantum system, one must demand that the following constraint hold: the mapping must be one-to-one and differentiable. Every space-time point (r, t, θ, φ) now maps onto exactly one point $(x, y, \theta', \varphi')$, without the emergence of cusp singularities.

The emergence of a non-trivial topology needs not be completely absurd, as long as no signals can be sent around much faster than the speed of light. This is the case at hand here.

Black emptiness: blue regions are the accessible part of space-time; dotted lines indicate identification.

The white sphere within is *not* part of space-time. Call it a 'vacuole'.



At given time t , the black hole is a 3-dimensional vacuole. The entire life cycle of a black hole is a vacuole in 4-d Minkowski space-time.

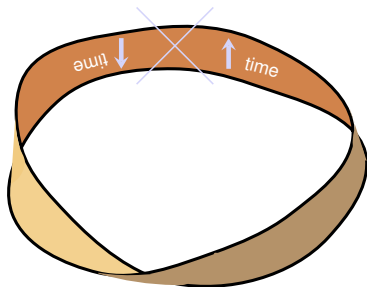
Space coordinates change sign at the identified points

– *and also time changes sign*

(Note: time stands still at the horizon itself).

Not that all u^\pm and p^\pm coordinates are odd when switching between antipodes. Therefore, only odd ℓ contribute in the spherical harmonic expansion.

A timelike Möbius strip



Draw a space-like closed curve:

Begin on the horizon at a point

$$r_0 = 2GM, \quad t_0 = 0, \quad (\theta_0, \varphi_0) .$$

Move to larger r values, then
travel to the antipode:

$$r_0 = 2GM, \quad t_0 = 0, \quad (\pi - \theta_0, \varphi_0 + \pi) .$$

You arrived at the same point,
so the (space-like) curve is closed.

Now look at the environment $\{dx\}$ of this curve. Continuously transport dx around the curve. The identification at the horizon demands

$$dx \leftrightarrow -dx, \quad dt \leftrightarrow -dt, .$$

So this is a Möbius strip, in particular in the time direction.

Note that it makes a T inversion and a CP inversion when going around the loop. CPT is preserved, **herewith correcting an error in earlier versions of these notes.**

There are no direct contradictions, but take in mind that

the local Hamiltonian density switches sign as well (see also slides 51 – 53).

This is not true for the *total* Hamiltonian adopted by distant observers. locally, near the horizon, this is the *dilaton operator*. That operator leaves regions *I* and *II* invariant, and does not flip sign along the loop. Also, the boundary condition, our “scattering matrix”, leaves this Hamiltonian invariant.

The *S*-matrix commutes with the Hamiltonian.

There are numerous treatises in the literature claiming solutions to the black hole information paradox, and about as many publications that dismiss these claims.

This author dismisses all claims, from both sides, that either ignore the gravitational back reaction of quantised excitations, or ignore the antipodal identification of points on the horizon – meaning that the horizon is a *projective 2-sphere*.

This concludes the most essential parts of the theory. If you have many burning questions, please contact g.thoof@uu.nl. Only if you are really interested in further details, problems and attacks aimed at solving them, consider PART 2, below.

PART 2, Further details and discussion

Discreteness of the microstates

Now let us describe the full Hilbert space of a black hole. Let us split it up:

BH: The (classical) background metric, for simplicity the Schwarzschild one, *filled with only soft particles*, both in-going and out-going, but now we keep only those particles that, at a given, fixed, time τ are close to the horizon, typically inside a region $2M_{\text{BH}} \leq r \leq 2M_{\text{BH}}(1 + \varepsilon)$, where ε is small but not very small, say $\varepsilon = 0.1$.

RU: The rest of the universe; all contributions of in- and out-particles which, at the time τ , have $r > 2M_{\text{BH}}(1 + \varepsilon)$.

The total mass-energy, of BH + RU, is conserved, but since the split is a function of time, the mass of the black hole is time dependent.

Let us take $\tau = 0$. According to Slide # 16, we then have $|x| = |y| \lesssim \sqrt{\varepsilon}$, or $|u^\pm| \lesssim 2GM\sqrt{2\varepsilon}$. The fact that u^\pm is taken to be inside this interval, implies that p^\mp is discrete:

$$p^\mp = \frac{\pi n^\mp}{2GM\sqrt{2\varepsilon}} = u^\mp \frac{\ell^2 + \ell + 1}{8\pi G}.$$

Both u^+ and u^- are constrained, so that the integers n^\pm are constrained:

$$|n^\pm| \leq \varepsilon \frac{(\ell^2 + \ell + 1)GM^2}{\pi^2} \equiv n_0(\ell) \varepsilon.$$

$u^+(\ell, m)$ does not commute with $u^-(\ell, m)$, so we take either n^+ or n^- to characterise the state at (ℓ, m) . The entire microstate can thus be characterised by the sequence

$$\{n^+(0, 0), \dots, n^+(\ell_{\max}, m_{\max})\}$$

(assuming some maximal values for (ℓ, m)).

The total number of states is (note that ℓ is odd):

$$N = \prod_{\ell, m} (2n_0(\ell) \varepsilon) = e^{\sum_{\ell=\text{odd}} (2\ell + 1) \log(n_0(\ell) \varepsilon)} = \mathcal{O}(e^{\ell_{\max}^2 \log \ell_{\max}}).$$

Note that our limits for the values of $u^\pm(\ell, m)$ for large ℓ can only indirectly be interpreted as particle positions since $u_{\ell, m}^\pm$ are partial wave amplitudes rather than coordinates.

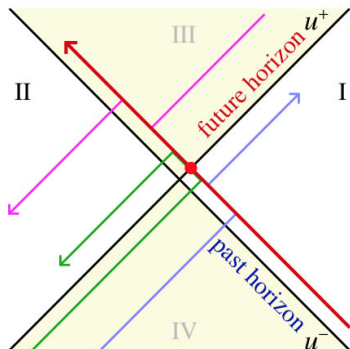
Assuming that $\ell_{\max} = \mathcal{O}(M_{\text{BH}}/M_{\text{Planck}})$, we recover the area law, but not the coefficient.

This topic must be further investigated.

Critical questions.

A bizarre sign switch

The effect of the gravitational footprint of an in-going particle (red line to upper left) onto the out-going particles (other coloured lines) is correctly accommodated for in this work. But there is an oddity, see our picture here of the allowed regions *I* and *II* and the unphysical regions *III* and *IV*.



The gravitational footprint moves the *pure states* of all out-particles combined in the light cone direction by an amount $\delta u^-(\theta, \varphi)$ in the same direction as the in-momentum $\delta p^-(\theta', \varphi')$.

This is a unitary transformation in the space of the combined wave functions only if all out-particles move by the same amount in the same direction, as is drawn here.

But this led to an apparently valid criticism:

This configuration does not appear to correspond to a valid solution of Einstein's equations for the local observer. What are we doing wrong?

It appears that this corresponds to a solution of Einstein's equations only if the momentum δp^- of the hard, in-particle flips its sign while passing through the future event horizon. At the same time, of course, the arrow of time switches sign.

Moreover, *those out-particles that are dragged right across the horizon (green line), do not seem to follow a geodesic at all, as regarded by the local observer.* Yet, of course, with our definition of the arrow of time, we have no choice, the particle has to continue its way as shown. What should a local observer say?

For the local observer, the diagram here does not show a classical solution, but it shows the action of an operator. The operator creates an extra in-particle (red line) that, in all states in regions I and II , shifts the geodesics of all out-particles by the same amount, in the same direction.

But if we represent the result of this operator as what happens later in time, then the fact that time runs backwards in region II leads to the impression that a discontinuity occurs on the other (future) horizon.

Operators should not be regarded as effects that agree with evolution equations; they represent what happens to a system if we make a change on the Cauchy surface, at some moment in time.

They subsequently affect the entire geodesics of the particles.

The change brought about here, has the effect of a positive δp^- particle in region I and at the same time the effect of a negative δp^- particle (an annihilated particle) in region II . Note that the operators u^\pm and p^\pm on the horizons all switch sign when passing from region I to II , since they are each other's antipodes, on the horizon.

An other way to formulate this sign problem was brought forward to me by E. Witten (private discussion):

If you quantise a field, you have the commutation rule:

$$[\phi(\vec{x}), \partial_0 \phi(\vec{x}')] = i\delta^3(\vec{x} - \vec{x}') .$$

If, in the transformation of region II to the antipodes of region I, the direction of time is inverted, so this rule gets a minus sign. This (again) implies that creation and annihilation operators for particles are interchanged.

This would affect the definition of the vacuum state as being the state where one can create particles while there are no particles to annihilate. If the antipodal domain of region I is (almost) empty, then in region II of the eternal Penrose diagram, the local observer would experience an *almost full* state, a state that is the counter part of the vacuum state.

The best way to handle this feature is to emphasise that, in our treatment, we should not use quantised fields. In the immediate vicinity of the horizons, we are using momentum and position operators that interact as single, *first quantised* particles, whose energies can be positive or negative (a negative energy particle represents something that was annihilated from the black hole configuration that we started off from – note that the *vacuum, or ground state* of a black hole is the state where the black hole is absent altogether).

It seems that we do not need to explicitly define the 'full' counterpart of the vacuum state, just like the vacuum state that we also do not use. Annihilating or creating particles is done with the same operators before and after the Bogolyubov transformation.

However, in case the 'totally full' state does require to be defined, we can replace the commutation relation on slide 51 by something like

$$[\phi(\vec{x}), \partial_0 \phi(\vec{x}')] = i(\frac{1}{2} E_{\max} - H) \delta^3(\vec{x} - \vec{x}') .$$

This is the same algebra as that of the angular momenta,

$$[L_1, L_2] = iL_3 ,$$

so that we get finite dimensional representations. Standard QFT is then the $\ell \rightarrow \infty$ limit of this system.

Finite-dimensional representations will be needed anyway to understand the emergence of holography, and it is also a desired feature in our theories concerning the cellular automaton interpretation of quantum mechanics.

There is no need to worry that the 'full' counterpart of the vacuum would generate tremendous gravitational fields; we declared only those particles to be considered whose gravitational fields are negligible.

The sign problem also emerges when considering the role of the Hartle-Hawking state, see Slide 57.

We must emphasise here that our treatment of the sign problem was inspired by pragmatism: we asked: which treatment leads most accurately to a unitary scattering matrix?, which lead to the arguments exposed in these three slides. Our aim must be to search for a matching treatment of quantum gravity as a whole. A skeptical reader may be left with lots of questions, so we expect these not to be the last words on this peculiar feature of black holes.

The BMS group.

In a development parallel to our work here, it has been argued that “conservation of information” for black holes can be understood by using the BMS group. It was claimed that this group generates an infinite class of conserved charges that must be held responsible for safeguarding the information processed by the black hole. These charges are associated to supertranslations, and as such must take the form of (θ, φ) -dependent momenta in the light cone direction. To the present author, the physical interpretation of these arguments is less transparent. The arguments seem to be very formal, and thus suspect. I would like to see how the BMS group generates a unitary S -matrix.

It is quite conceivable that what is called BMS charges coincides with the momentum distributions that are central in our discussions. The functions we call $p^\pm(\theta, \varphi)$ do correspond to the entire momentum entering (p^-) or leaving (p^+) at given solid angle $\Omega(\theta, \varphi)$ as accumulated over all time. So situating these “charges” on a projective square at \mathcal{J}^\pm makes sense.

The black hole mass

Other question often asked: how does the mass of the black hole respond on the absorbed and emitted particles?

This is actually very easy. The energy (the sum of the numbers κ at every value of (ℓ, m)), is exactly conserved. To see how this works, arrange all in- and out- particles into three classes:

- (i) The particles coming in when still far away from the future horizon, which are the firewall transforms of the out-particles that are still very close to the past event horizon, ready to emerge at some later era,
- (ii) The soft particles in the parts of regions I and II that are in the neighbourhood, but not too close to any of the horizons, and
- (iii) The particles moving out at a safe distance from the black hole, they are the firewall transforms of the particles that have arrived at a very close distance from the future event horizon.

For reader(s) still concerned about the sign problem: note that energy is only well-defined for the outside observer, so we do not need to worry about a sign switch of energy when crossing the horizon.

The total energy, black hole plus these particles, is strictly conserved. But the particles of group (i) (the very late particles) and group (iii) (the very early particles) are so far away that we should not count those as part of the black hole mass/energy. The particles of group (ii) however, are so close to the horizon that these must be considered as part of the black hole.

We have to postulate that the entire black hole mass consists of the contributions of these particles.

Clearly, this mass is time dependent, as the distinction of a particle's classes is time dependent.

Note however, that a black hole is always surrounded by the cloud of entangled Hartle-Hawking particles.

The complete set of black hole states

When a local observer sees a vacuum, it means that the black hole is in the *Hartle-Hawking state*,

$$|\Omega\rangle = C \sum_{E,n} e^{-\frac{1}{2}\beta E} |E, n\rangle_I |E, n\rangle_{II} .$$

It is seen as a *vacuum* $|\Omega\rangle$ for the local observer, but contains states $|E, n\rangle$ in regions *I* and *II* for the global observer. C is a normalisation constant, and n represents all quantum numbers other than the energies E of the particle states – as seen by the global observer.

The *local* observer may now use his creation and annihilation operators to go to other states. All these states remain pure because of the antipodal identification, but we lose the entanglement of the above *HH* state.

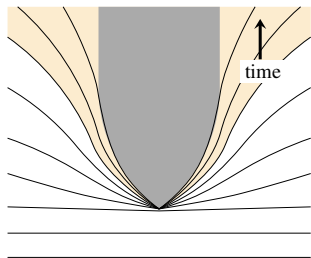
Here also, as in slides 51 – 53, the sign switch problem emerges: time runs backwards in region *II*, so, while for the local observer, this is a vacuum, where the energies left and right add up to zero, the same energies add up for the distant observer. We must accept that the antipodal identification requires $i \leftrightarrow -i$.

The moment of collapse and the moment of the final explosion

How does the antipodal identification get started when matter collapses into a black hole?

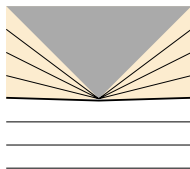
How does it end after the hole ended its life through evaporation?

This is a hard problem because at those moments the background metric is non stationary. Energy and momentum are ill defined.

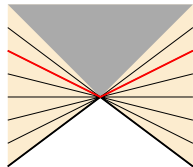


1) Realistic coordinates, with equal-time lines

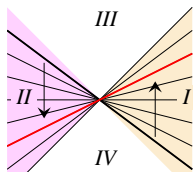
Penrose diagrams:



2) situation at $t = 0$ showing Cauchy surfaces



3) situation at $t \gg 0$



4) antipodal identification

To obtain straight Cauchy surfaces, we need to invert time in region II, the antipode.

and now the grav. shift can be accounted for.

Previous slide:

We describe here a metric with time-dependent source. the above was a suggestion for taking the back reaction into account, also during collapse. From picture (4) onwards, we propose to work just as in the “eternal” case.

When inspected in picture (1), we see that

The antipodal identification also implies an inversion in the time direction.

The entire inside region (grey) is removed,

so that only pure particle states can arise, also with gravitational back reaction. As in the eternal case, we replace the energetic in-particles at the past horizon by their footprints: the soft out-particles at the future event horizon.

The time-reverse of this, the final evaporation of the black hole, must be handled in exactly the same way.

More on the horizon algebra.

In my publications between ~ 1985 and ~ 1996 , it was emphasised that the amplitudes produced by the mechanism of the gravitational back reaction, are strongly related to those of string theory. Perhaps it is string theory, although it seems more like a string theory with purely imaginary α' . The mathematical similarity originates simply from the fact that, as in string theory, we end up with algebraic relations of particle coordinates on a two-dimensional surface.

The string world sheet is replaced by the point of intersection of the future and the past event horizons. In- and out-going particles resemble the vertex insertions on a closed-string world sheet. Thus, one might expect an algebra akin to the Virasoro algebra. Yet I want to derive the black hole horizon algebra with utmost care, since the physical circumstances are quite different. If there is a relation, we need to find it, but also focus on differences.

In what follows we start producing some commutation rules, but calculations were only just begun.

For the time being, replace the spherical horizon(s) by an infinite, flat Rindler space-time. Replace the spherical harmonics $Y_{\ell m}(\Omega)$ by plane waves $e^{i\tilde{k}\cdot\tilde{x}}$, where the flat, transverse coordinates \tilde{x} replace $\Omega = (\theta, \varphi)$ and \tilde{k} replaces the "wave numbers" (ℓ, m) .

We consider the case $\tilde{k}^2 \ll M_{\text{Planck}}^2$ (This may later be relaxed).

Write the blue equations in slide 26 as the Fourier transforms

$$u^{\pm}(\tilde{x}) = \frac{1}{2\pi} \int d^2\tilde{k} e^{\pm i\tilde{k}\cdot\tilde{x}} \hat{u}^{\pm}(\tilde{k}),$$

$$p^{\pm}(\tilde{x}) = \frac{1}{2\pi} \int d^2\tilde{k} e^{\pm i\tilde{k}\cdot\tilde{x}} \hat{p}^{\pm}(\tilde{k}),$$

2-vectors are indicated by a tilde ($\tilde{}$). The inverse equations are

$$\hat{u}^{\pm}(\tilde{k}) = \frac{1}{2\pi} \int d^2\tilde{x} e^{\mp i\tilde{k}\cdot\tilde{x}} \hat{u}^{\pm}(\tilde{x}),$$

$$\hat{p}^{\pm}(\tilde{k}) = \frac{1}{2\pi} \int d^2\tilde{x} e^{\mp i\tilde{k}\cdot\tilde{x}} \hat{p}^{\pm}(\tilde{x}),$$

Signs \pm are chosen for later convenience. We omit the carets ($\hat{}$) as their locations should be clear from the notation.

$u^{\pm}(\tilde{x})$ is gravitational displacement at \tilde{x} caused by *all* p_i^{\pm} :

Important: the *momenta* p^\pm are now handled as distributions, built up from deltas where particles enter;
 the positions $u^\pm(\tilde{x})$ just mark the location of particles with respect to their horizons. Labelling the particles by an index i , we take:

$$p^\pm(\tilde{x}) \equiv \sum_i p_i^\pm \delta^2(\tilde{x} - \tilde{x}_i) \quad ; \quad u^\pm(\tilde{x}_i) = u_i^\pm ,$$

$$u^-(\tilde{x}) = 8\pi G \sum_i f(\tilde{x} - \tilde{x}_i) p_i^- = 8\pi G \int d^2\tilde{x}' f(\tilde{x} - \tilde{x}') p^-(\tilde{x}') ;$$

Green function obeys: $\tilde{\partial}^2 f(\tilde{z}) = -\delta^2(\tilde{z}) .$

Our commutation rules are now:

$$[u_i^\pm, u_j^\pm] = [u_i^\pm, p_j^\pm] = [p_i^\pm, p_j^\pm] = 0 , \quad [u_i^\pm, p_j^\mp] = i \delta_{ij} ,$$

and the algebra turns into:

$$u^-(\tilde{k}) = \frac{8\pi G}{\tilde{k}^2} p^-(\tilde{k}) \quad ; \quad [u^\pm(\tilde{x}'), p^\mp(\tilde{x})] = i \delta^2(\tilde{x}' - \tilde{x}) .$$

$$[u^\pm(\tilde{k}), p^\mp(\tilde{k}')] = i \delta^2(\tilde{k} - \tilde{k}') \quad ; \quad [p^-(\tilde{k}), p^+(\tilde{k}')] = \frac{i\tilde{k}^2}{8\pi G} \delta^2(\tilde{k} - \tilde{k}') .$$

$$u^+(\tilde{k}) = -\frac{8\pi G}{\tilde{k}^2} p^+(\tilde{k}) \quad ; \quad \tilde{\partial}^2 u^\pm(\tilde{x}) = \pm 8\pi G p^\pm(\tilde{x}) .$$

Note Newton's G in denominator. This algebra is "non-perturbative" in gravity.

For future need:

$$[u^\pm(\tilde{x}), p^\mp(\tilde{k})] = \frac{i}{2\pi} e^{\pm i\tilde{k}\cdot\tilde{x}}, \quad [u^\pm(\tilde{k}), p^\mp(\tilde{x})] = \frac{i}{2\pi} e^{\pm i\tilde{k}\cdot\tilde{x}}.$$

Because of the delta functions in the previous slide, the modes for the different harmonic waves \tilde{k} all decouple. Formally, a generic quantum state (replacing Fock space for the Standard Model + perturbative gravity) is the product of states: $|u^+(\tilde{k})\rangle$ for all allowed values of \tilde{k} .

Equivalently, one can label these states as $|u^-(\tilde{k})\rangle$ or $|p^+(\tilde{k})\rangle$ or $|p^-(\tilde{k})\rangle$. These modes are like the one-particle states in QFT, but not quite, they label *matter waves*. Even so, we just have single wave functions $\psi(u^+)$ for a single coordinate u^+ , at each value for \tilde{k} .

The wave functions $\psi(p^-)$, one for each value of \tilde{k} , are the Fourier transforms of $\psi(u^+)$, see slide 32. $\psi(p^\pm)$ is the same as $\psi(u^\pm)$, after scaling the variables p^\pm to u^\pm , see previous slide.

It is important not to confuse the *wave modes* indicated by their *mode number* \tilde{k} with the actual momentum this state might have. This will be indicated by the operator $\tilde{P} \equiv \int d^2\tilde{k} \tilde{p}(\tilde{k})$, which equals $\sum_i \tilde{p}_i$ for all particles i . Thus, like p^\pm , \tilde{p} is a distribution.

In order to calculate how each mode \tilde{k} contributes an amount $\tilde{p}(\tilde{k})$ to the total transverse momentum, let us displace all harmonics by the same (infinitesimal) distance \tilde{a} . Then:

$$p^+(\tilde{k}) e^{i\tilde{k}\cdot\tilde{x}} \rightarrow p^+(\tilde{k}) e^{i\tilde{k}\cdot(\tilde{x}+\tilde{a})}, \quad \text{or: } p^+ \rightarrow p^+ e^{i\tilde{k}\cdot\tilde{a}}; \quad \delta p^+ = i\tilde{k} \cdot \tilde{a} p^+ \equiv i\tilde{a} \cdot \tilde{P},$$

and, noting that u^\pm and p^\pm are all in the same \tilde{k} mode, we find that this mode's contribution to the momentum operator \tilde{P} is:

$$\begin{aligned} \tilde{P} &= \tilde{k} \left(p^+ \frac{\partial}{\partial p^+} + \frac{1}{2} \right) = -\tilde{k} \left(p^- \frac{\partial}{\partial p^-} + \frac{1}{2} \right) = -i\tilde{k} (p^+ u^- + \frac{1}{2}i) \\ &= \tilde{k} \left(u^+ \frac{\partial}{\partial u^+} + \frac{1}{2} \right) = -\tilde{k} \left(u^- \frac{\partial}{\partial u^-} + \frac{1}{2} \right) = -i\tilde{k} (u^- p^+ - \frac{1}{2}i) \end{aligned}$$

in agreement with the relations between u^\pm and p^\pm on slide 63.

What are the eigen values of \tilde{P} ? To understand the question, we return to Hermitian operators (the plane waves are complex, hence non Hermitian).

Split up $e^{\pm i\tilde{k}\cdot\tilde{x}} = \cos(\tilde{k}\cdot\tilde{x}) \pm i \sin(\tilde{k}\cdot\tilde{x})$.

Write $p^\pm(-\tilde{k}) = p^{\pm*}(\tilde{k})$, and

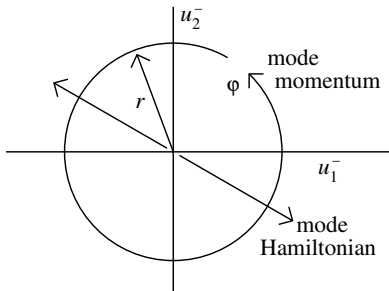
$$p^\pm(\tilde{k}) e^{\pm i\tilde{k}\cdot\tilde{x}} + p^\pm(-\tilde{k}) e^{\mp i\tilde{k}\cdot\tilde{x}} \equiv p_1^\pm \cos(\tilde{k}\cdot\tilde{x}) + p_2^\pm \sin(\tilde{k}\cdot\tilde{x}),$$

$$p^\pm(\tilde{k}) = p_1^\pm \pm ip_2^\pm; \quad u^\pm(\tilde{k}) = \frac{1}{2}(u_1^\pm \pm iu_2^\pm).$$

The $\frac{1}{2}$ is there to keep commutation rules as $[u_\alpha^\pm, p_\beta^\pm] = i\delta_{\alpha\beta}$, where $\alpha, \beta = 1, 2$. This gives the transverse momentum operator contribution of the two modes \tilde{k} and $-\tilde{k}$ together:

$$\begin{aligned} \tilde{P} &= -\frac{1}{2}\tilde{k}(u_1^- - iu_2^-)(p_1^+ + ip_2^+) + h.c. = \tilde{k}(u_1^- p_2^+ - u_2^- p_1^+) \\ &= i\tilde{k} \frac{\partial}{\partial \varphi}. \end{aligned}$$

$$\tilde{P} = -\frac{1}{2}\tilde{k}(u_1^- - iu_2^-)(p_1^+ + ip_2^+) + h.c. = \tilde{k}(u_1^- p_2^+ - u_2^- p_1^+) \\ = i\tilde{k}\frac{\partial}{\partial\varphi} .$$



This generates a rotation in (u_1^-, u_2^-) space. The Hamiltonian, $H = u^- p^+ - \frac{1}{2}i = u_1^- p_1^+ + u_2^- p_2^+ - i$ generates a dilation (see slide 36)

The wave function for these two modes can be written as $\psi(u_1^-, u_2^-) = \psi(r, \varphi)$ (in polar coordinates), see Figure.

\tilde{P} generates a *rotation* while H generates a *dilation*
(actually, a Lorentz transformation, if we take u_i^+ and u_i^- together)

Note that φ is periodic, and therefore the eigenvalues of \tilde{P} are given by integers $n \in \mathbb{Z}$:

$$\tilde{P} = \tilde{k} n .$$

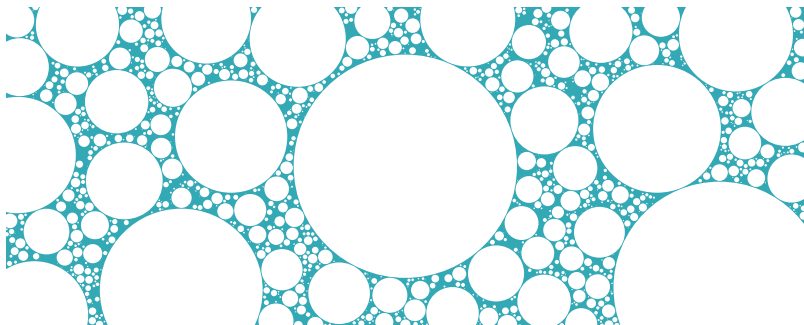
This illustrates that \tilde{k} itself is *not* to be interpreted as transverse momentum; *transverse momentum is a multiple of \tilde{k}* ,

and similarly, (ℓ, m) are *not* to be interpreted as angular momenta. Rather, they are spherical harmonic mode numbers.; angular momentum is a multiple of ℓ .

We end with a speculation. It was introduced in version 3 of these slides.
Ignore it if you only want transparent physics.

Virtual black holes and space-time foam (Summary)

Virtual black holes must be everywhere in space and time. Due to vacuum fluctuations, amounts of matter that can contract to become black holes, must occur frequently. They also evaporate frequently, since they are very small. This produces small vacuoles in the space-time fabric. How to describe multiple vacuoles is not evident. The emerging picture could be that of “space-time foam”:



Work is still in progress. More than happy to discuss these ideas with like-minded colleagues.

The author benefited from discussions with P. Betzios, N. Gaddam, O. Papadoulaki, S. Mathur, L. Susskind, J. Maldacena and many others.

See: G. 't Hooft, [arxiv:1612.08640 \[gr-qc\]](https://arxiv.org/abs/1612.08640) + references there;
[arxiv:1804.05744 \[gr-qc\]](https://arxiv.org/abs/1804.05744);

See also:

P. Betzios, N. Gaddam and O. Papadoulaki, The Black Hole S-Matrix from Quantum Mechanics, JHEP 1611, 131 (2016), [arxiv:1607.07885](https://arxiv.org/abs/1607.07885).

S.W. Hawking, M.J. Perry and A. Strominger, Superrotation Charge and Supertranslation Hair on Black Holes, [arXiv:1611.09175 \[hep-th\]](https://arxiv.org/abs/1611.09175)

N. Sanchez and B. Whiting, Quantum Field Theory and the Antipodal Identification of Black Holes, Nucl.Phys. **B283** (1987) 605.

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