Problem 1, Lorentz transformations of electric and magnetic fields

We have that

$$F^{\mu'\nu'} = L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} F^{\mu\nu},$$

where,

$$F^{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_2 & iE_2 & iE_3 & 0 \end{pmatrix}.$$

Note that we use the convention from the lecture notes, in which $x^\mu = (x, y, z, ict)$.

(1) Consider an ordinary rotation around the z-axis. Show that the magnetic and electric fields transform as ordinary vectors under such a rotation.

(2) Consider a boost along the x-axis with velocity v. Show that we have the following transformation, mixing electric and magnetic fields: ($\gamma = 1/\sqrt{1-\beta^2}$, $\beta = v/c$)

$$\begin{aligned} B'_1 = B_1, \quad B'_2 &= \gamma(B_2 + \beta E_3), \quad B'_3 = \gamma(B_3 - \beta E_2), \\ E'_1 = E_1, \quad E'_2 &= \gamma(E_2 - \beta B_3), \quad E'_3 = \gamma(E_3 + \beta B_2). \end{aligned}$$

Recall that a Lorentz boost along the x-axis is given by the transformation

$$x'^1 = \gamma(x^1 + i\beta x^4) \quad y' = y \quad z' = z \quad x'^4 = \gamma(x^4 - i\beta x^1).$$

Consider now an inertial system **I** and the Lorentz boosted system **I′** (i.e. **I′** not rotated relative to **I**), moving with velocity $\overrightarrow{v}$ relative to **I**. The formula generalizing the above formula is:

$$\begin{aligned} \overrightarrow{B} &= \gamma\overrightarrow{B'} + \frac{\overrightarrow{v}}{v^2}(\overrightarrow{v}\cdot\overrightarrow{B'})(1-\gamma) + \gamma\frac{\overrightarrow{v}}{c}\times\overrightarrow{E'}, \\ \overrightarrow{E} &= \gamma\overrightarrow{E'} + \frac{\overrightarrow{v}}{v^2}(\overrightarrow{v}\cdot\overrightarrow{E'})(1-\gamma) - \gamma\frac{\overrightarrow{v}}{c}\times\overrightarrow{B'}. \end{aligned}$$

(You can try to prove this formula, but it is not part of the problem) Assume that $\overrightarrow{E}'$ and $\overrightarrow{B}'$ are constant and different from zero.

(3) Find the condition that $\overrightarrow{E}'$ and $\overrightarrow{B}'$ have to satisfy in order that there exists a $\overrightarrow{v}$ such that $\overrightarrow{B} = 0$, and find the corresponding $\overrightarrow{v}$ expressed in terms of $\overrightarrow{E}'$ and $\overrightarrow{B}'$.

Problem 2

'Constant acceleration part I' of the enclosed exercises.

Problem 3, The Eötvös experiment

In the lecture it was explained how Eötvös found a way of testing the Equivalence Principle (i.e. $M_{grav} = M_{inert}$) by checking the misalignment of the forces acting on two objects. The gravitational force on the surface of the Earth is

$$\vec{F}_g = -G_N M_\bigoplus M_{grav} \frac{\vec{r}}{r^3},$$

while the centrifugal force is

$$\vec{F}_\omega = M_{inert}\omega^2 \left(\vec{r} - \frac{(\vec{\omega} \cdot \vec{r})\vec{\omega}}{\omega^2} \right),$$

where ω is Earth's angular velocity. Can you derive more carefully the measurable misalignment between the forces $\vec{F}^{(i)} = \vec{F}_g^{(i)} + \vec{F}_\omega^{(i)}$ of two objects:

$$\alpha = \frac{|\vec{F}^{(1)} \wedge \vec{F}^{(2)}|}{|\vec{F}^{(1)}||\vec{F}^{(2)}|} \approx ?$$

under the (justified) assumption that the gravitational force is much stronger than the centrifugal one?

Problem 4, Constant acceleration, part II

(0) Recall your results from last week, where we did the first part of this exercise

Now consider a coordinate system moving ”along” with the accelerated particle and *rigid* in the sense that distances measured with standard rods at rest in the system are constant in time (which, in the co-moving frame, is identified with the proper time of the accelerated particle).

In part I we found the transformation from the inertial system I, where the particle started at rest at $t = 0$, to another inertial system I', where the accelerated system (particle) is at rest at proper time τ and coordinate time $t' = 0$. Now, we *define* a coordinate system (w^0, w, y, z) by

$$w^0 = \tau, \quad w = x', \quad y = y', \quad z = z',$$

where we have to set $t' = 0$. Such a coordinate system will move along with the accelerated particle. It is very special that we have a transformation from Minkowski space to this co-moving frame. In general this is not possible. In the following we will ignore y and z. Thus we have the relation between (x^0, x) and (w^0, w):

$$\begin{aligned} x &= \frac{c^2}{g}\left[\cosh\left(\frac{gw^0}{c^2}\right) - 1\right] + w\cosh\left(\frac{gw^0}{c^2}\right), \\ x^0 &= \frac{c^2}{g}\sinh\left(\frac{gw^0}{c^2}\right) + w\sinh\left(\frac{gw^0}{c^2}\right). \end{aligned}$$

(w^0, w) are called Rindler coordinates (R).

(1) Consider two spacetime points infinitesimally seperated. Let they have coordinates (x^0, x) and $(x^0 + dx^0, x + dx)$. Denote the corresponding coordinates in R by (w^0, w) and $(w^0 + dw^0, w + dw)$. Show that

$$ds^2 \equiv dx^2 - (dx^0)^2 = dw^2 - (1 + gw/c^2)^2(dw^0)^2$$

and conclude that the Rindler coordinates are indeed rigid (lengths are constant).

(2) Show that the point with fixed coordinate w in R as seen from I, performs a hyperbolic motion with velocity

$$v = \frac{gx^0/c}{\sqrt{(1 + gw/c^2)^2 + (gx^0/c^2)^2}} \quad (= c\tanh\frac{gw^0}{c^2})).$$

(3) Using the results from part (1), show that an observer at rest in I, with spatial coordinate x can send signals to any other observer at rest in I. Consider a point $w > 0$. Show that there are points x which can never send signals to an observer in rest at R with coordinate w. Characterize the region of space time which cannot send signals to any observer at rest in R. We say that the system R has a horizon: There are regions of space-time which can receive signals from R, but cannot send signals to R.

Further Hints for this exercise:

$$\cosh[\text{arcsinh}x] = \sqrt{1+x^2}$$
$$\tanh[\text{arcsinh}x] = \frac{x}{\sqrt{1+x^2}}$$

Problem 5, Rotating coordinate system

In order to find the line element of a uniformly rotating reference frame, we can start from flat space in cylindrical coordinates,

$$ds'^2 = -c^2 dt'^2 + d\rho'^2 + dz'^2 + \rho'^2 d\phi'^2,$$

and perform the following spatial transformation for axis of rotation z:

$$\rho = \rho', \qquad z = z', \qquad \phi = \phi' + \omega t.$$

Here ω is the constant angular velocity of rotation.

1. What is the line element ds^2 of the rotating coordinate system?
2. What would be the circumference of a circle in terms of the coordinate ρ as measured by an observer in the rotating frame? Discuss the physical meaning of the case when $\rho \geq \frac{c}{\omega}$.

Hint: An observer in a given space-time measures the time interval between two events as $d\tau^2 = g_{00} dt^2$ *and the space interval as* $dl^2 = \left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}}\right) dx^i dx^j$.

Problem 6, Coordinate transformations

1. How would the product of a covariant and a contravariant vector, $A^\mu B_\mu$, transform under general coordinate transformations?

2. What about the quantity $g_{\mu\nu}g^{\nu\rho} = \delta_\mu^\rho$?

3. Can you show that the derivative of a covariant vector, $\partial_\mu A_\nu$, does not transform as a tensor? Show that for linear coordinate transformations (i.e. Lorentz transformations) $\partial_\mu A_\nu$ is indeed a tensor.

Problem 7, Transformation rule of a density

Consider a fully antisymmetric tensor $g_{\mu\nu\alpha\beta}$ in four dimensional space time.
(1) Argue that $g_{\mu\nu\alpha\beta}$ has indeed one independent component and therefore can be written as $g_{\mu\nu\alpha\beta} = \omega\epsilon_{\mu\nu\alpha\beta}$, where $\epsilon_{\mu\nu\alpha\beta}$ is the Levi-Cevita symbol, which is defined in the notes.
(2) Using Leibnitz formula for the determinant, show that ω transform as a (scalar) density,

$$\tilde{\omega}(u) = \det\left(\frac{\partial x}{\partial u}\right)\omega(x(u)).$$

Leibnitz formula:

$$\det A = \epsilon_{\alpha\beta\gamma\delta}A_0^\alpha A_1^\beta A_2^\gamma A_3^\delta.$$

(3) Using Leibnitz' formula, show how the Levi-Cevita symbol transforms under coordinate transformation. An object that transforms in this way is called a *tensor density* (of weight 1, in this case). We will see later that the determinant of the metric transforms as a tensor density of weight -2.
(4) Prove identity (5.18) of the lecture notes:

$$D_\alpha\omega = \partial_\alpha\omega - \Gamma^\mu_{\mu\alpha}\omega.$$

Problem 8, Transformation rule of an affine connection field

Do the exercise on page 20 of the lecture notes.

Problem 9

'The energy momentum tensor' of the enclosed exercises.

Problem 10

'Derivation of the geodesic equation' of the enclosed exercises.

Problem 11

'Geodesics in Rindler space' of the enclosed exercises.

Problem 12

'Geodesics on the rotating disk' of the enclosed exercises.

Problem 13

The metric for the three-sphere in coordinates $x^\mu = (\psi, \theta, \phi)$ can be written

$$ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2).$$

(a) Calculate the Christoffel connection coefficients.
(b) Calculate the Riemann tensor, Ricci tensor, Ricci scalar and Einstein tensor.
(c) Show that

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})$$

A space for which this last equation is satisfied is called a *maximally symmetric space*. In such a space, the curvature is the same everywhere and the same in every direction. Hence, if one knows the curvature in one point of the space, it is known everywhere.

Problem 14

A vector field is a mapping that assigns to every point in space-time an object in its tangent space. In a curved space-time it is not straightforward to compare two vectors at different points. Only if two vectors are elements of the same tangent space, one can add, subtract, or take the dot product of two vectors. Think of two vectors on the sphere: If one of them lives on the equator and the other one on the north pole, there is no way to compare them. However, there is a way to transform a vector from one tangent space to the other along a given path. The concept of transporting a vector from one point in space to another point, while keeping it 'constant' is known as *parallel transport*. If one wants to keep a vector constant along a curve, $x^\mu(\lambda)$, in flat space, one wants to keep the components of the vector field constant along the curve:

$$\frac{d}{d\lambda}V^\mu = \frac{dx^\nu}{d\lambda}\frac{\partial V^\mu}{\partial x^\nu} = 0.$$

In a curved space time, one replaces the partial derivative by the covariant derivative, and then defines parallel transport of the vector V^μ along the curve $x^\mu(\lambda)$ to be the requirement that the covariant derivative of V^μ along the path vanishes:

$$\frac{D}{d\lambda}V^\mu \equiv \frac{dx^\nu}{d\lambda}D_\nu V^\mu = 0,$$

where we have defined the **directional covariant derivative**, $D/d\lambda$. The concept of parallel transport is defined for general tensors.

Recall that a geodesic is the curved space generalization of a straight line. A straight line is a path that parallel transports its own tangent vector.
(1) Show that this definition of a straight line gives indeed the geodesic equation.

Another definition of a straight line is the path of extremal distance between two points. Therefore, the geodesic equation can be derived by minimizing the action

$$S[x(\lambda)] = \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}.$$

(2) Convince yourself that these two definition are only equivalent if one chooses the Christoffel connection. (use the proper time as parameter along your path, such that $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -1$).

Assume we have a metric of the form:

$$ds^2 = da^2 + g_{bb}(a) db^2.$$

(3) Show that only $\Gamma^b_{aa}, \Gamma^a_{bb}$, and $\Gamma^b_{ab} = \Gamma^b_{ba}$ are different from zero and given by

$$\Gamma^b_{aa} = g^{bb}\frac{dg_{bb}}{da}, \quad \Gamma^a_{bb} = -\frac{1}{2}\frac{dg_{bb}}{da}, \quad \Gamma^b_{ab} = \Gamma^b_{ba} = \frac{1}{2}g^{bb}\frac{dg_{bb}}{da}.$$

(4) Choose polar coordinates in the Euclidian plane and consider a vectorfield $V = (V^r, V^\theta)$ and a path along a circle with radius r_0: $x(\lambda) = (r(\lambda), \theta(\lambda)) = (r_0, \theta_0 + \lambda(\theta_1 - \theta_0)), \lambda \in [0, 1]$ Let $V(\lambda = 0) \equiv (V^r_0, V^\theta_0)$ sit at the point (r_0, θ_0). Parallel transport this vector along the path given. What is $V(\lambda = 1)$, (which is an element of the tangent space at (r_0, θ_1)).

(5) Consider a vectorfield $V = (V^\phi, V^\theta)$ on the unit sphere, where the metric is given by

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2, \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi[.$$

Parameterize a path along a curve with constant $\theta = \theta_0$: $x(\lambda) = (\phi(\lambda), \theta(\lambda)) = (\phi_0 + \lambda(\phi_1 - \phi_0), \theta_0), \lambda \in [0, 1]$ Let $V(\lambda = 0) \equiv (V^\phi_0, V^\theta_0)$ sit at the point (ϕ_0, θ_0). Parallel transport this vector along the path given. What is $V(\lambda = 1)$, (which is an element of the tangent space at (ϕ_1, θ_0)).

(6) Check that the norm of this vector is conserved after parallel transport.

(7) A cone is parameterized by $x = a\,u\,\cos\theta, y = a\,u\,\sin\theta, z = u$, where $u \geq 0, \theta \in [0, 2\pi[$. One can reparameterize the cone in terms of s and θ, where s is the distance to the tip of the cone. In this parametrization the metric has the form we use in this exercise. Calculate the metric.

(8) Calculate parallel transport around a circle surrounding the tip of the cone (constant s).

Problem 15

Consider the expanding universe metric:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

(1) Find the geodesic equations. (Either by explicitly constructing the Christoffel symbols, or by varying the action $S = \int d\lambda \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$)
(2) Using the geodesic equations and the expression for ds^2, show that for a photon moving along the x-axis, the coordinate-time as function of the parameter λ, which parameterizes the path, is given by

$$\frac{dt}{d\lambda} = \frac{\omega_0}{a},$$

where ω_0 is a constant. Defining $p^\mu = dx^\mu/d\lambda$, show that the energy of the photon, $E = p_\mu U^\mu$, with U^μ the four velocity, is for a comoving observer equal to $E = \omega_0/a$. Can you explain why this is called the *cosmological redshift*?

Problem 16

'The Riemann tensor', part (1)-(3), of the enclosed exercises.

Problem 17

The Exercise at page 36 of the lecture notes.

Problem 18

'The energy momentum tensor' of the enclosed exercises.

Problem 19

Assume that we are in a space, where the Minkowski metric is slightly perturbed:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x).$$

(1) Consider coordinate transformations of the type: $x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x)$. Show that the metric transforms as:

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x) + D_\mu \xi_\nu(x) + D_\nu \xi_\mu(x).$$

$h_{\mu\nu}$ has ten independent components. The transformation above is a gauge transformation. If one wants to fix the gauge we have to choose the four ξ^μ. This implies that there have to be 6 gauge independent variables. In this exercise we want to find these 6 quantities. The fundamental theorem of vector calculus tells us that we can decompose any vector field into the sum of a part with vanishing divergence (transverse vector) and a part with vanishing curl.

$$A_i = A_i^v + A_i^c = A_i^v + \partial_i A^s, \quad \partial_i A_i^v = 0, \quad \nabla \times \overrightarrow{A}^c = 0,$$

where A^s is a scalar quantity. Therefore one can write the perturbation $h_{\mu\nu}$ as:

$$h_{00} = h_{00}, \tag{1}$$
$$h_{0i} = B_i + \partial_i S, \tag{2}$$
$$h_{ij} = \frac{\delta_{ij}}{3} h + (\partial_i \partial_j - \frac{\delta_{ij}}{3} \nabla^2) \tilde{h} + (\partial_i V_j + \partial_j V_i) + h_{ij}^{TT}, \tag{3}$$
$$\tag{4}$$

where h_{00}, h, S, and $\tilde{h}$ are scalars; B_i and V_i are transverse vectors, and h_{ij}^{TT} is a transverse traceless tensor (graviton).

(2) Argue that the number of degrees of freedom are equal on both sides of the equation signs in (1), (2), and (3).

(3) Using that $\xi_\mu = (\xi_0, \xi_i^v + \partial_i \xi^s)$, show that the transformation rules under a coordinate transformation for this new defined quantities are given by

$$\begin{aligned}
h_{00} &\to h_{00} + 2\partial_0 \xi_0, \\
h &\to h + \nabla^2 \xi^s, \\
S &\to S + \xi_0 + \partial_0 \xi^s, \\
\tilde{h} &\to \tilde{h} + 2\xi^s, \\
B_i &\to B_i + \partial_0 \xi_i^v, \\
V_i &\to V_i + \xi_i^v, \\
h_{ij}^{TT} &\to h_{ij}^{TT}.
\end{aligned}$$

(4) Show that there are two scalars one vector and one tensor, which are gauge independent by constructing them explicitly.
(5) Can you show that the scalar quantities are equal to $-GM/r$ in the Newtonian limit of general relativity? This statement is now gauge independent. (For more information about the Newtonian limit of GR see the notes from Carroll: http://arxiv.org/abs/gr-qc/9712019).