Exercise sheet 1: 24 Feb + 3 Mar

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before Friday 6 Mar, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Polar coordinates.** The polar coordinates (r, θ) in the Euclidean x, y-plane are given by $(x, y) = (r \cos\theta, r \sin\theta)$.
   (a) Derive the metric g_{ij} of the Euclidean plane in polar coordinates.
   (b) What are the Christoffel symbols?
   (c) Show that the Laplacian $\Delta f = g^{ij} D_i D_j f$ of a scalar function f in polar coordinates coincides with the familiar one from calculus.
   (d) Write down the geodesic equation for $\ddot{r}$ and $\ddot{\theta}$ and show that they are solved by straight lines in the x, y-plane.

2. **Determinant of the metric.** Let g denote the determinant of the metric $g_{\mu\nu}$.
   (a) Show that in three dimensions g is given by $g = \frac{1}{6}\epsilon^{\mu\nu\kappa}\epsilon^{\alpha\beta\gamma} g_{\mu\alpha} g_{\nu\beta} g_{\kappa\gamma}$, where $\epsilon^{\mu\nu\kappa}$ is the three-dimensional Levi-Civita symbol (i.e. totally antisymmetric and $\epsilon^{123} = 1$).
   (b) Show that $\epsilon^{\mu\nu\kappa}/\sqrt{|g|}$ transforms as a tensor under a coordinate transformation $x^\mu \to x^{\mu'}$.
   (c) Show that $\Gamma^\alpha_{\alpha\beta} = \partial_\beta \log\sqrt{|g|}$ (summation convention is implied here).

3. **Conformally flat metrics.** Let f be a positive function and $g_{\mu\nu}(x) = f(x)^2 \eta_{\mu\nu}$ be a metric in d dimensions. Such a metric is called conformally flat.
   (a) How do the light-cones look like?
   (b) Show that $\Gamma^\mu_{\nu\lambda} = \delta^\mu_\lambda \partial_\nu \log f + \delta^\mu_\nu \partial_\lambda \log f - \eta_{\nu\lambda}\partial^\mu \log f$.
   (c) Show that $R = -\frac{1}{f^2}\left(2(d-1)\partial^\mu\partial_\mu \log f + (d-2)(d-1)\partial^\mu log f\, \partial_\mu log f\right)$.
   (d) Show that we can write the metric of the unit 2-sphere, $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$, in the conformally flat form $ds^2 = \cosh^{-2} x(dx^2 + d\phi^2)$ by changing coordinates $x = \log(\tan(\theta/2))$.
   (e) Use part (c) to calculate the curvature R of the unit 2-sphere.

4. **Bianchi identity.**
   (a) Derive the Jacobi identity
   $$[[D_\alpha, D_\beta], D_\gamma] + [[D_\beta, D_\gamma], D_\alpha] + [[D_\gamma, D_\alpha], D_\beta] = 0$$
   by expanding the commutators.
   (b) Given a vector field V^μ. Show that $[D_\mu, D_\nu]V^\lambda = R^\lambda{}_{\rho\mu\nu} V^\rho$.
   (c) Use (a) and (b) to derive the Bianchi identity
   $$D_\alpha R^\mu{}_{\kappa\beta\gamma} + D_\beta R^\mu{}_{\kappa\gamma\alpha} + D_\gamma R^\mu{}_{\kappa\alpha\beta} = 0.$$