Exercise sheet 3: 24 Mar + 26 Mar

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 27 Mar, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Electromagnetism coupled to gravity.** The action for general relativity coupled to a matter field is of the form

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_m\right), \tag{1}$$

where $\mathcal{L}_m$ is a Lagrangian corresponding to the matter field.

(a) Show that when we vary S with respect to $g_{\mu\nu}$ we get the Einstein equations with stress-energy tensor given by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g_{\mu\nu}} = -2\frac{\delta\mathcal{L}_m}{\delta g_{\mu\nu}} - g^{\mu\nu}\mathcal{L}_m. \tag{2}$$

You may use that $\delta\int d^4x\sqrt{-g}R = \int d^4x\sqrt{-g}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})\delta g^{\mu\nu}$.

(b) For electromagnetism without sources $\mathcal{L}_m$ is given by

$$\mathcal{L}_m(A_\mu, g_{\mu\nu}, J^\mu) = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength. Derive the corresponding $T_{\mu\nu}$.

(c) Show that the inhomogeneous Maxwell equation $D_\mu F^\mu_\nu = 0$ (with $J^\mu = 0$) is equivalent to $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$.

(d) Assume we are in the static spherical symmetric case $ds^2 = -A(r)\,dt^2 + B(r)dr^2 + r^2d\Omega^2$ with only an electric field in the r-direction, i.e. $F_{0r} = -F_{r0} = E(r)$ and other components vanish. Show that the inhomogeneous Maxwell equation implies that

$$E(r) = \frac{Q\sqrt{AB}}{4\pi r^2} \tag{4}$$

for some constant Q.

(e) Show that $T_{\mu\nu} = \frac{Q^2}{32\pi^2r^2}\mathrm{diag}(-\frac{A}{r^2}, \frac{B}{r^2}, -1, -\sin^2\theta)$.

2. **Killing vectors.** A continuous symmetry of a metric is characterized by a Killing vector field ξ_μ, which satisfies $D_\mu\xi_\nu + D_\nu\xi_\mu = 0$. Show that $\xi_\mu\dot{x}^\mu$ is conserved along a geodesic $\lambda \to x(\lambda)$.

3. **Falling into a Reissner-Nordström black hole.** The Reissner-Nordström metric with charge Q is given by

$$ds^2 = -Adt^2 + A^{-1}dr^2 + r^2 d\Omega^2 \tag{5}$$

with $A = 1 - 2M/r + Q^2/(4\pi r^2)$.

(a) Show that we can rewrite the metric as

$$ds^2 = -Adv^2 + 2dvdr + r^2 d\Omega^2 \tag{6}$$

in terms of a co-moving time $v = t + f(r)$ (the analogue of (8.6) in the lecture notes) satisfying

$$\frac{df}{dr} = \frac{1}{A(r)}. \tag{7}$$

(b) Solve (7). Hint: Use that $A(r) = (1 - \frac{r^+}{r})(1 - \frac{r^-}{r})$.

(c) Draw the light-cone structure in the (v, r)-plane.

(d) Let ξ^μ be the vector field corresponding to $\partial/\partial t$, i.e. time translation. Show that ξ^μ is a killing vector field and therefore the "energy" $E = -\xi_\mu \dot{x}^\mu$ is conserved for an infalling particle.

(e) Show that for a radially infalling (massive) particle we have

$$\dot{r}^2 + A(r) = E^2, \tag{8}$$

where $\dot{r} = dr/d\tau$ is a derivative with respect to the eigentime parameterizing the geodesic.

(f) We can interpret (8) as the equation for a particle with energy E^2 in a 1-dimensional potential $A(r)$. Draw this potential (and indicate where the horizons $r^\pm$ are). What would we expect to happen for a particle coming from $r = \infty$?

(g) Show that

$$\frac{dv}{d\tau} = \frac{E \pm \sqrt{E^2 - A}}{A}. \tag{9}$$

where the sign depends on whether the particle is ingoing or outgoing. What happens to $\dot{v}$ near the horizons? Argue that the coordinates brake down when the particle encounters the horizon r^- for the second time.

(h) Show that if we change coordinates to (u, r) with $u = v - 2f(r)$, that the coordinate singularity is removed. Therefore we have extended our space-time metric to a new region. What does the metric look like in these coordinates?

(i) What happens for a radially infalling massless particle?