# Exercise sheet 4: 31 Mar

## NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 3 Apr, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Killing vectors again.** Suppose that in a particular coordinate system $\{x^\mu\}$ the metric is independent of one of the coordinates, say $x^0$. Show that then $\xi^\mu = \delta^\mu_0$ is automatically a Killing vector.

2. **Energy extraction from a rotating black hole.** The Kerr solution for the metric of a rotating black hole is given by

$$ds^2 = -\frac{\Delta}{Y}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{Y}(adt - (r^2+a^2)d\phi)^2 + \frac{Y}{\Delta}dr^2 + Y d\theta^2, \qquad (1)$$

with $Y = r^2 + a^2\cos^2\theta$, $\Delta = r^2 + a^2 - 2Mr$ and $a$ is the angular momentum (per unit of mass). The singularity is located at $Y = 0$ and the horizons at $\Delta = 0$.

   (a) What are the positions $r_-$ and $r_+$ of the inner and outer horizon?

   (b) Show that $\xi^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial\phi)^\mu$ are Killing vectors.

   (c) The region outside of the outer horizon where the killing vector $\xi^\mu$ becomes space-like is called the *ergosphere*. Calculate the (boundary of the) ergosphere and make a sketch. Show that the ergosphere disappears when $a = 0$, i.e. when the metric becomes that of the Schwarzschild black hole.

   (d) Why does this imply that observers in the ergosphere cannot remain stationary? Indeed, show that for any time-like curve in the ergosphere we must have $\dot\phi > 0$ (hint: write out $g_{\mu\nu}\dot x^\mu \dot x^\nu < 0$ and use that $\dot t > 0$).

   (e) Denote the energy of a particle with mass $m$ by $E = -p^\mu\xi_\mu$, where $p^\mu = m\dot x^\mu$ is its 4-momentum. Recall that $E$ is conserved along geodesics. Show that a particle falling freely into the ergosphere always has $E > 0$, while a particle which is created inside the ergosphere can have $E < 0$.

   (f) Show that when a particle with energy $E_0$ decays into two particles with energies $E_1$ and $E_2$, that by energy-momentum conservation we must have $E_0 = E_1 + E_2$.

   (g) Argue that it is possible to shoot a particle with energy $E$ into the ergosphere and that another comes out with an energy larger than $E$.

   (h) Let $L = p^\mu\psi_\mu$ denote the angular momentum of a particle. It can be shown that any particle entering the horizon at $r_+$ must have $E - \frac{a}{r_+^2+a^2}L > 0$. Use this to argue that when we extract energy from a black hole we necessarily reduce its angular momentum (you may use that the angular momentum $J = aM$ of the black hole changes by an amount equal to $L$). Therefore we can only repeat the process until $a = 0$.