Exercise sheet 5: 7 Apr

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 10 Apr, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Nearly global coordinates for the Reissner-Nordström black hole.** Recall that the metric of a black hole with charge Q is given by

$$ds^2 = -Adt^2 + A^{-1}dr^2 + r^2 d\Omega^2 \tag{1}$$

with $A = 1 - 2M/r + Q^2/(4\pi r^2)$. We take the coordinates u and v as in exercise 3 on sheet 3,

$$v = t + f(r), \quad u = t - f(r), \tag{2}$$

with

$$f(r) = r + \frac{r_+^2}{r_+ - r_-}\log|r - r_+| - \frac{r_-^2}{r_+ - r_-}\log|r - r_-|. \tag{3}$$

(a) Show that in these coordinates the metric is given by

$$ds^2 = -A(r)dudv + r^2 d\Omega^2, \tag{4}$$

where r is viewed as a function $r(u, v)$. Where is the outer horizon ($r = r_+$) in terms of the coordinates u and v? What part of space-time do the coordinates describe?

(b) The analogues of the Kruskal coordinates (see section 6 in the lecture notes) are

$$x = \exp\left(\frac{r_+ - r_-}{2r_+^2}v\right), \quad y = \exp\left(-\frac{r_+ - r_-}{2r_+^2}u\right). \tag{5}$$

Show that the metric becomes

$$ds^2 = \left(\frac{2r_+^2}{r_+ - r_-}\right)^2 \frac{|r - r_-|^{1+(r_-/r_+)^2}}{r^2}\exp\left(-\frac{r_+ - r_-}{r_+^2}r\right)dxdy + r^2 d\Omega^2, \tag{6}$$

where $r(x, y)$ is implicitly defined by

$$xy = \exp\left(\frac{r_+ - r_-}{r_+^2}r\right)(r - r_+)\frac{r - r_-}{|r - r_-|^{1+(r_-/r_+)^2}}. \tag{7}$$

Show that the metric is non-degenerate in the entire x, y-plane.

(c) Define the coordinates x_P and y_P as in section 12 of the lecture notes,

$$x = \tan(\frac{\pi}{2}x_P), \quad y = \tan(\frac{\pi}{2}y_P). \tag{8}$$

Draw the part of space-time with $r > r_-$ in the x_P, y_P-plane. Where are the horizons? Draw some $r =$ constant and $t =$ constant curves.

(d) The metric is still singular at the inner horizon $r = r_-$ in these x_P, y_P coordinates. Define now new coordinates $x_1,\ y_1$ by

$$x = \exp\left(\frac{r_+ - r_-}{2r_-^2} v\right), \quad y = \exp\left(-\frac{r_+ - r_-}{2r_-^2} u\right). \tag{9}$$

Show that now the metric is regular at the horizon $r \to r_-$.

(e) We get a regular metric everywhere in the Penrose diagram for Reissner-Nordstrom, if we take the Penrose coordinates by combining the coordinates (5) and (9) (assuming $r_+ > r_-$)

$$\tan x_P = x + x_1\ , \qquad \tan y_P = y + y_1\ . \tag{10}$$

Explain why this works, and find the domain for these coordinates to define the maximal extension of this space-time.

(f) How can one construct x_P and y_P for the Kerr-Newman solution?