Exercise sheet 7: 28 Apr

NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 4 May, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **Surface gravity of the Kerr-Newman black hole.** Recall that the Kerr-Newman metric is given by

$$ds^2 = -\frac{\Delta}{Y}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{Y}(adt - (r^2+a^2)d\phi)^2 + \frac{Y}{\Delta}dr^2 + Y d\theta^2, \qquad (1)$$

with $Y(r) = r^2 + a^2\cos^2\theta$ and

$$\Delta = r^2 + a^2 - 2Mr + \frac{Q^2}{4\pi} = (r - r_+)(r - r_-). \qquad (2)$$

   (a) Show that if we replace $\phi = \tilde{\phi} + \frac{a}{r_+^2+a^2}t$ the metric (1) for constant θ and $\tilde{\phi}$ and close to $r = r_+$ is given by

$$ds^2 = Y(r_+)\left(-\frac{(r-r_+)(r_+-r_-)}{(r_+^2+a^2)^2}dt^2 + \frac{dr^2}{(r-r_+)(r_+-r_-)}\right) \qquad (3)$$

   up to higher order terms in $(r - r_+)$.

   (b) Define a new coordinate $\rho = \rho(r)$ such that the metric takes the form of Rindler space-time

$$ds^2 = -(\kappa\rho)^2 dt^2 + d\rho^2. \qquad (4)$$

   Show that κ (the "surface gravity") is given by

$$\kappa = \frac{1}{2}\frac{r_+ - r_-}{r_+^2 + a^2}. \qquad (5)$$

2. **Free fall in Rindler space-time.** The 2-dimensional Rindler metric is given by

$$ds^2 = -(\kappa\rho)^2 d\tau^2 + d\rho^2, \qquad (6)$$

for a constant κ.

   (a) Show that the equation of motion for a massive particle in Rindler spacetime is given by

$$\frac{d^2\rho}{d\tau^2} = -\rho\left(\kappa^2 + \frac{2}{\rho^2}(\frac{d\rho}{d\tau})^2\right). \qquad (7)$$

   (b) For what value of ρ is τ a normalized (or proper) time-coordinate? What is the gravitational field at that point as felt by an observer at rest? This is by definition the surface gravity (corresponding to the time τ).

   (c) Given a value $\rho_0 > 0$, how should we rescale τ for it to be normalized at $\rho = \rho_0$? What is the surface gravity corresponding to the new time coordinate? How does it depend on ρ_0?

We see that the definition of surface gravity depends on the chosen time coordinate. However, for a stationary asymptotically flat space-time metric (like Schwarzschild and Kerr-Newman) there exists a natural choice of time coordinate corresponding to the proper time at spatial infinity (which we used in the previous exercise).

3. Show that the period in Euclidean time is $2\pi/\kappa$.