# Exercise sheet 8: 12 May

## NS-TP431: Introduction to black holes (spring 2009)

The exercises are to be handed in during exercise class or brought to Timothy Budd in office MG 418 before 18 May, 5pm. The exercises will not be graded, but we expect all exercises to be done and we will try to give feed-back when necessary.

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1. **The Unruh effect.** In this exercise we will check various calculations from the lecture notes.

   (a) Derive the Klein-Gordon equation in the Rindler coordinates,
   $$\left((\rho\partial_\rho)^2 - \partial_\tau^2 + \rho^2(\tilde{\partial}^2 - m^2)\right)\Phi = 0, \tag{1}$$
   directly from the Klein-Gordon action
   $$S[\Phi] = -\int d^4x\sqrt{-g}(\partial^\mu\Phi\partial_\mu\Phi + m^2\Phi^2). \tag{2}$$

   (b) Show that the operator $a_2$ defined by equation (17.12),
   $$a_2(\tilde{k},\omega) = \int_{-\infty}^{\infty}\frac{dk^3}{\sqrt{2\pi k^0}}a(\vec{k})e^{i\omega\ln\left(\frac{k^3+k^0}{\mu}\right)}, \tag{3}$$
   with $k^0(\vec{k}) = \sqrt{(k^3)^2+\mu^2}$ and $\mu^2 = \tilde{k}^2 + m^2$, satisfies the commutation relations
   $$[a_2(\tilde{k},\omega), a_2^\dagger(\tilde{k}',\omega')] = \delta^2(\tilde{k}-\tilde{k}')\delta(\omega-\omega'). \tag{4}$$

   (c) Prove equation (17.24) from the lecture notes,
   $$\begin{aligned} H &= \int_{-\infty}^{\infty} d\omega\,\omega\int d^2\tilde{k}\,a_2^\dagger(\tilde{k},\omega)a_2(\tilde{k},\omega) \qquad (5)\\ &= \int_0^\infty d\omega\,\omega\int d^2\tilde{k}\left(a_I^\dagger(\tilde{k},\omega)a_I(\tilde{k},\omega) - a_{II}^\dagger(\tilde{k},\omega)a_{II}(\tilde{k},\omega)\right) = H_I - H_{II}. \end{aligned}$$
   The minus sign here reflects the fact that in wedge $II$ time runs backwards!

   (d) Use the expression for ${}_M\langle\Omega|\,\mathcal{O}\,|\Omega\rangle_M$ from equation (17.31) to show that in the state $|\Omega\rangle_M$ the expected number of particles $\langle N_{\omega,\tilde{k}}\rangle$ with energy $\omega$ and wave-vector $\tilde{k}$ is given by the Bose-Einstein statistics
   $$\langle N_{\omega,\tilde{k}}\rangle = \frac{1}{e^{2\pi\omega}-1}. \tag{6}$$