Introduction to General Relativity
exercises

8. As it was explained during the lectures, the notion of symmetry (and antisymmetry) is an invariant property of tensors with an arbitrary number of indices. Consider the useful and widely used definitions for the completely symmetrized (and antisymmetrized) part of an arbitrary tensor:

$$T_{(\alpha_1\alpha_2\ldots\alpha_n)} = \frac{1}{p!}\sum_P T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{1}$$

$$T_{[\alpha_1\alpha_2\ldots\alpha_n]} = \frac{1}{p!}\sum_P (-1)^{|P|} T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{2}$$

Here the summation is for all permutation P of the indices, $(-1)^{|P|}$ is the parity, -1 for odd, 1 for even permutations.

How many independent components do these tensors have (for arbitrary n) ?

How many independent components do these tensors have if the dimension d is also arbitrary?

Use the result to show that for the generic case (that is when the tensor is not constrained by any symmetry property) the tensor $T_{\alpha_1\alpha_2\ldots\alpha_n}$ itself is **not** the sum of the completely symmetric and antisymmetric part for $n > 2$.

Of course the process for tensors with purely upper indices the procedure is identical. Can you symmetrize in mixed indices ? That is, e.g. is $T^\mu_\nu + T^\nu_\mu$ a tensor ?

9. You are given a tensor $K^{\alpha\beta}$. How can you test whether it's a direct product of two vectors $K^{\alpha\beta} = A^\alpha B^\beta$ or not ? Is your test independent of the choice of coordinates ?