Introduction to General Relativity
exercises

1. Prove that the stress-energy tensor (1.24) obeys the equation

$$\partial_\mu T_{\mu\nu} = F_{\nu\lambda} J_\lambda, \tag{1}$$

making use of Maxwell's equations (1.16), (1.17). Explain this result for the fourth and the spacelike components of the equation. Interpreting T_{00} as energy density and T_{0i} as momentum density, explain how J_ν takes up energy and/or momentum.

2. With the help of the relations

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta \tag{2}$$

and the covariant transformation rule of the gradient:

$$\frac{\partial}{\partial x_i} = \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} \tag{3}$$

write down the Schrodinger equation in polar coordinates for e.g. the Hydrogen atom: $V = e^2/r$. How does the Laplacian emerge from the "curved" coordinates (r, θ, ϕ) ?

3. Consider the general case of a particle folllowing the arbitrary trajectory $x^\nu(\tau)$ in its rest frame (τ is the proper time). Let $L_{\mu\nu}(\tau')$ be the transformation from the rest frame of the particle at $\tau = 0$ to the rest frame at $\tau = \tau'$. Let $\vec{g}(\tau)$ be the acceleration of the particle as determined in its own reference frame at time τ. Determine the differential equation which $\Lambda_{\mu\nu}(\tau_1)$ obeys. Hint: Study the two successive infinitesimal transformation by which the rest frame is obtained for a particle (An infinitesimal transformation is of the form $\delta_{\mu\nu} + \epsilon_{\mu\nu}$ where $\epsilon_{\mu\nu}$ is antisymmetric). Clearly $L_{\mu\nu}(0) = \delta_{\mu\nu}$. Solve the differential equation for the case that the particle moves along the z-axis with constant acceleration g, to show that formula (3.11) holds for the relation between the coordinates of the comoving grid and the inertial observer.

4. Consider the motion of a cannonball shot vertically upwards with the velocity $v_0 << c$ until it falls back to the Earth. There are clocks, both on the cannonball and on the ground.

Assuming *only* the rules of special relativity, but ignoring the possible effects of the gravitational field, calculate the proper time of the cannonball relative to that of the observer standing on the Earth, in the inertial frame of the Earth to the first non-trivial order in v_0^2/c^2.

Now turn to a coordinate grid moving with the cannon ball and repeat the calculation. Note the sign difference in these two results. There is a fundamental difference between the two frames as the cannon ball's watch may speed up due to the gravitational potential. As is indicated in the lecture notes, its proper time τ speeds up as $\mathrm{d}\tau = \mathrm{dt}(1 + \mathrm{gz})$, $\mathrm{d}t$ being the time interval on Earth, while an observer in the inertial frame seen from the cannon ball experiences no gravitational field at all, so in this inertial frame, time slows down only due to the special relativistiv effect.

Calculate the time difference due to the gravitational field in the Earth's frame of reference, and show that the twin 'paradox' now disappears; the two calculations agree.

5. Form "natural" units of mass, length and time out of the physical constants $\hbar$, G and c. Calculate their values.

6. Estimate the Bohr radius of a "gravitational" atom, e.g. two neutrons bound by their gravitational attraction in their lowest energy state.

7. Consider the star operation on a two-form in four dimensional Minkowski space given by

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{4}$$

Show that, when it is applied on the electromagnetic field strength tensor, then $E \to B$ and $B \to -E$ for the three-vectors of the electric and magnetic field.

Show that if $F_{\mu\nu}$ is a solution of the free-space Maxwell equations, so is $\star F$ and also $e^{\star a}F \equiv F\cos\alpha + \star F\sin\alpha$ for arbitrary α. (Note that the latter transformation is called "duality rotation").

8. As it was explained during the lectures, the notion of symmetry (and antisymmetry) is an invariant property of tensors with an arbitrary number of indices. Consider the useful and widely used definitions for

the completely symmetrized (and antisymmetrized) part of an arbitrary tensor:

$$T_{(\alpha_1\alpha_2\ldots\alpha_n)} = \frac{1}{p!}\sum_P T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{5}$$

$$T_{[\alpha_1\alpha_2\ldots\alpha_n]} = \frac{1}{p!}\sum_P (-1)^{|P|} T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{6}$$

Here the summation is for all permutation P of the indices, $(-1)^{|P|}$ is the parity, -1 for odd, 1 for even permutations.

How many independent components do these tensors have (for arbitrary n) ?

How many independent components do these tensors have if the dimension d is also arbitrary?

Use the result to show that for the generic case (that is when the tensor is not constrained by any symmetry property) the tensor $T_{\alpha_1\alpha_2\ldots\alpha_n}$ itself is **not** the sum of the completely symmetric and antisymmetric part for $n > 2$.

Of course the process for tensors with purely upper indices the procedure is identical. Can you symmetrize in mixed indices ? That is, e.g. is $T^\mu_\nu + T^\nu_\mu$ a tensor ?

9. You are given a tensor $K^{\alpha\beta}$. How can you test whether it's a direct product of two vectors $K^{\alpha\beta} = A^\alpha B^\beta$ or not ? Is your test independent of the choice of coordinates ?

10. Prove (5.5) in the lecture notes: show that two successive transformations of that form produce one of the same form.

11. Check (5.8a).

12. Derive the expression (5.27) for the curvature starting from Eq. (5.31) as a *definition* of the Riemann tensor.

13. Suppose, that we define the covariant derivative abstractly as the linear map, which obeys Leibnitz rule: (5.15), commutes with contraction: (5.16), and acts as ordinary derivative on scalar fields (5.17). Show (or argue) that the difference of two derivative operators gives rise to a tensor C^c_{ab}. (That is $(\nabla_\mu - \tilde{\nabla}_\mu)A_\nu = C^\alpha_{\mu\nu}A_\alpha$. (Hint: The difference at

point P depends only on the value of the vector field A_μ at that point and it is also a linear transformation.) Show that it is symmetric in its lower indices, if we require that $\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$ vanishes on scalar fields. (In the general case its antisymmetric part is called torsion.) Show, using (5.9) for the covariant derivative, that this is equivalent of the symmetricity of the quantity $\Gamma^\alpha_{\mu\nu}$

14. Derive the Bianchi identity (5.30) in the coordinate frame where

$$\Gamma^\mu_{\nu\alpha} = 0 \ , \tag{7}$$

following the hints in the text after (5.30).

15. Prove the following symmetry properties (5.26) and (5.29)of the Riemann curvature tensor, supposing (5.27) to be the definition. Argue that, for the latter, we may also use the special frame where $\Gamma^\mu_{\nu\alpha}$ vanishes locally.

16. Using the symmetry properties given in Chapter 5, derive the number of *mutually independent* components of the Riemann tensor.

    When we will know how to raise and lower indices we will find one more symmetry property of the Riemann tensor:

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \tag{8}$$

    How many independent components does the Riemann tensor have then? How many does it have in d dimensions?

17. Consider a manifold M with metric $g_{\mu\nu}$. Take a point $x \in M$ fixed. Show that there exists a special coordinate system in which the connections coefficients are zere at that point.

$$\Gamma^\alpha_{\mu\nu}(x) = 0 \tag{9}$$

    Such a coordinate system is called a *geodesic coordinate system.* Note: if the above property holds for the whole M in one and the same coordinate system, then the manifold is called *affine flat.* Show that we may furthermore choose our coordinates such that the metric is of the form:

$$g_{\mu\nu}(x) = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1). \tag{10}$$

Show that (1) (for all values of the indices α, μ, ν) implies

$$\partial_\alpha g_{\mu\nu}(x) = 0. \tag{11}$$

Property (1), (2) and (3) defines the notion of a *local inertial frame* at x. Which transformations preserve the properties of a local inertial frame ?

Choose a local inertial frame about x. Let the coordinates of x be the origin of the frame. Show that in a neighbourhood of x the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}(\partial_\alpha\partial_\beta g_{\mu\nu})x^\alpha x^\beta \tag{12}$$

Compute the Riemann curvature, the Ricci tensor and the Ricci scalar at the point x. For simplicity, assume $\eta_{\mu\nu} = \delta_{\mu\nu}$.

18. A space purports to be 3-dimensional with coordinates x, y, z and the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - (\frac{3}{13}dx + \frac{4}{13}dy + \frac{12}{13}dz)^2. \tag{13}$$

Show that it is really a two-dimensional space and find two new coordinates χ, η for which the line element takes the form

$$ds^2 = d\chi^2 + d\eta^2 \tag{14}$$

. Hint: Calculate the volume element or density function for this space to convince yourself, that the coordinates are linearly dependent so that we can omit one and perform a linear (orthogonal?) transformation to obtain the 2-dimensional Euclidean line element (6).

19. Show that a contraction of a vector with the projection tensor: $g_{\mu\nu} + \epsilon u_\mu u_\nu$ projects the vector into the 3-surface orthogonal to the vector u_μ, where $\epsilon = 1$ if u_μ is timelike, $\epsilon = -1$ if u_μ is spacelike, and $|u^\mu u_\mu| = 1$ for these cases. Show that there is no unique projection orthogonal to a lightlike or null vector.

20. Consider the sphere S^2: $(x^0, x^1, x^2) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. A coordinate patch on $S^2 \backslash \{(1,0,0)\}$ is given by the map $x^a(\xi^i)$, $(a = 0, 1, 2, \quad i = 1, 2)$:

$$x^0 = \frac{(\xi^j)^2 - 1}{(\xi^j)^2 + 1} \tag{15}$$

$$x^i = \frac{2\xi^i}{(\xi^j)^2 + 1}, \quad j = 1, 2 \tag{16}$$

which corresponds to a stereographic projection from the North Pole onto a plane through an equator. (Note that there is a summation in $(\xi^j)^2$ for j.) Take $\xi^1 = r\cos\phi$, $\xi^2 = r\sin\phi$ and show that the embedded metric $g_{ij} = \partial_i x^a \partial_j x^b \delta_{ab}$ is given by

$$ds^2 = \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\phi^2) \tag{17}$$

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Show that the stereographic projection is only invertible on $S^2 \backslash \{(1,0,0)\}$ and determine its inverse. How many other coordinate patches are needed to cover S^2 ? Give these other patches and also the coordinate transformations that relates two patches.

Calculate the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ corresponding to the metric above. (Hint: Use the trick that the Euler Lagrange equations corresponding to the variation of $x^\mu(t)$ coming from the Lagrangian: $\frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}$ is the geodetic eqution from where it is easy to read off the Christoffel symbols.) If you used the hint, prove its statement.

Show that the curves $r(u) = \tan(u/2)$ and $\phi = \Phi$ (constant) are geodesics.

Calculate the transformation of an arbitrary vector that gets paralell transported along the curve: $r = R$ (constant), $\phi_i = \Phi$, $\phi_f = \Phi + \alpha$.

For which value of R the above curve is geodesic ?

Calculate the surface of the region $0 < r < R$.

21. Consider a smooth nonzero real function (scalar field) Ω on a manifold M with the metric $g_{\mu\nu}$. The metric $\Omega^2 g_{\mu\nu}$ is said to arise from $g_{\mu\nu}$ via a *conformal transformation*.

    Write down the tensor $C^\alpha_{\mu\nu}$ which may be defined by the equation

$$\bar{\nabla}_\mu v_\nu = \nabla_\mu v_\nu - C^\alpha_{\mu\nu} v_\alpha \tag{18}$$

where $\bar{\nabla}_\mu$ is the Levi-Civita covariant derivative corresponding to $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

Show (or argue) that a metric is a positive multiple of another one at a point x if and only if the casual structure is the same for both at the point x. That is the notions spacelike, timelike and null concerning the norm of a vector at point x are invariant under conformal transformations and conversely: if they are invariant at one point then the metric is necessarily a positive multiple of the original metric at that point.

Show that the equation

$$v^\nu \nabla_\nu v^\mu = \alpha v^\mu \tag{19}$$

is the geodesic equations for the tangent $v^\mu = \frac{dx^\mu}{d\lambda}$ of a curve. I.e. it can be reparametrized so that the right hand side disappears.

Show that null geodesics are invariant under conformal transformations.

If a field $\bar{\Psi} = \Omega^s \Psi$ is a solution of a field equation corresponding to the metric $\Omega^2 g_{\mu\nu}$ as long as Ψ solves the field equation corresponding to $g_{\mu\nu}$ then it is called *conformally invariant.* Here, s is the conformal dimension of the field Ψ.

Show that the generalized Klein Gordon equation (for curved space) $g^{\mu\nu}\nabla_\mu \nabla_\nu \Phi = 0$ is conformally invariant only in two dimensions.

Show that the source free, curved space Maxwell equations ($\partial \to \nabla$) are conformally invariant in any dimensions.

Show that the conservation law $\nabla_\mu T^{\mu\nu} = 0$ for the stress-energy tensor of a physical system is conformally invariant only if $T^{\mu\nu}$ is symmetric and traceless.

Calculate the values of s in all three cases.