General Relativity exercises

1. Prove that the stress-energy tensor (1.24) obeys the equation

$$\partial_\mu T_{\mu\nu} = F_{\nu\lambda} J_\lambda \ , \tag{1}$$

making use of Maxwell's equations (1.16), (1.17). Explain this result for the 0th and the spacelike components of the equation. Interpreting T_{00} as energy density and T_{0i} as momentum density, explain how J_ν takes up energy and/or momentum.

2. With the help of the relations

$$x = r\sin\theta\cos\phi \ , \quad y = r\sin\theta\sin\phi \ , \quad z = r\cos\theta \tag{2}$$

and the covariant transformation rule of the gradient:

$$\frac{\partial}{\partial x_i} = \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j} \tag{3}$$

write down the Schrodinger equation in polar coordinates for e.g. the Hydrogen atom: $V = -e^2/r$. How does the Laplacian emerge from the "curved" coordinates (r, θ, ϕ)?

3. Consider the general case of a particle following the arbitrary trajectory $x^\nu(\tau)$ in its initial rest frame (τ is the proper time). Let $L_{\mu\nu}(\tau')$ be the transformation from the rest frame of the particle at $\tau = 0$ to its rest frame at $\tau = \tau'$. Let $\vec{g}(\tau)$ be the acceleration of the particle as determined in its own reference frame at time τ. Determine the differential equation which $L_{\mu\nu}(\tau)$ obeys. Hint: Study the two successive infinitesimal Lorentz transformations by which the rest frame is obtained for a particle (An infinitesimal transformation is of the form $\delta_{\mu\nu} + \epsilon_{\mu\nu}$, where $\epsilon_{\mu\nu}$ is antisymmetric). Clearly, $L_{\mu\nu}(0) = \delta_{\mu\nu}$. Solve the differential equation for the case that the particle moves along the z-axis with constant acceleration g, to show that formula (3.11) holds for the relation between the coordinates of the comoving grid and the inertial observer.

4. Consider the motion of a cannon ball shot vertically upwards with the velocity $v_0 << c$ until it falls back to the Earth. There are clocks, both on the cannon ball and on the ground.

Assuming *only* the rules of special relativity, but ignoring the possible effects of the gravitational field, calculate the proper time of the cannon ball relative to that of the observer standing on the Earth, in the inertial frame of the Earth, to first non-trivial order in v_0^2/c^2.

Now turn to a coordinate grid moving with the cannon ball and repeat the calculation. Note the sign difference in these two results. There is a fundamental difference between the two frames: in the first, the cannon ball's watch may speed up due to the gravitational potential. As is indicated in the lecture notes, its proper time τ speeds up as $d\tau = dt(1+gz)$, dt being the time interval on Earth, while an observer in the inertial frame seen from the cannon ball experiences no gravitational field at all, so in this frame, time slows down only due to the special relativistic effect.

Calculate the time difference due to the gravitational field in the Earth's frame of reference, and show that the twin 'paradox' now disappears; the two calculations agree.