Introduction to General Relativity
exercises

17. Consider a manifold M with metric $g_{\mu\nu}$. Take a point $x \in M$ fixed. Show that there exists a special coordinate system in which the connections coefficients are zere at that point.

$$\Gamma^{\alpha}_{\mu\nu}(x) = 0 \tag{1}$$

Such a coordinate system is called a *geodesic coordinate system*. Note: if the above property holds for the whole M in one and the same coordinate system, then the manifold is called *affine flat*. Show that we may furthermore choose our coordinates such that the metric is of the form:

$$g_{\mu\nu}(x) = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1). \tag{2}$$

Show that (1) (for all values of the indices α, μ, ν) implies

$$\partial_{\alpha} g_{\mu\nu}(x) = 0. \tag{3}$$

Property (1), (2) and (3) defines the notion of a *local inertial frame* at x. Which transformations preserve the properties of a local inertial frame ?

Choose a local inertial frame about x. Let the coordinates of x be the origin of the frame. Show that in a neighbourhood of x the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}(\partial_{\alpha}\partial_{\beta} g_{\mu\nu}) x^{\alpha} x^{\beta} \tag{4}$$

Compute the Riemann curvature, the Ricci tensor and the Ricci scalar at the point x. For simplicity, assume $\eta_{\mu\nu} = \delta_{\mu\nu}$.

18. A space purports to be 3-dimensional with coordinates x, y, z and the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - (\frac{3}{13}dx + \frac{4}{13}dy + \frac{12}{13}dz)^2. \tag{5}$$

Show that it is really a two-dimensional space and find two new coordinates χ, η for which the line element takes the form

$$ds^2 = d\chi^2 + d\eta^2 \tag{6}$$

. Hint: Calculate the volume element or density function for this space to convince yourself, that the coordinates are linearly dependent so that we can omit one and perform a linear (orthogonal?) transformation to obtain the 2-dimensional Euclidean line element (6).

19. Show that a contraction of a vector with the projection tensor: $g_{\mu\nu} + \epsilon u_\mu u_\nu$ projects the vector into the 3-surface orthogonal to the vector u_μ, where $\epsilon = 1$ if u_μ is timelike, $\epsilon = -1$ if u_μ is spacelike, and $|u^\mu u_\mu| = 1$ for these cases. Show that there is no unique projection orthogonal to a lightlike or null vector.