Introduction to General Relativity
exercises

20. Consider the sphere S^2: $(x^0, x^1, x^2) : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. A coordinate patch on $S^2 \backslash \{(1,0,0)\}$ is given by the map $x^a(\xi^i)$, $(a = 0,1,2, \quad i = 1,2)$:

$$x^0 = \frac{(\xi^j)^2 - 1}{(\xi^j)^2 + 1} \tag{1}$$

$$x^i = \frac{2\xi^i}{(\xi^j)^2 + 1}, \quad j = 1,2 \tag{2}$$

which corresponds to a stereographic projection from the North Pole onto a plane through an equator. (Note that there is a summation in $(\xi^j)^2$ for j.) Take $\xi^1 = r\cos\phi, \quad \xi^2 = r\sin\phi$ and show that the embedded metric $g_{ij} = \partial_i x^a \partial_j x^b \delta_{ab}$ is given by

$$ds^2 = \frac{4}{(1+r^2)^2}(dr^2 + r^2 d\phi^2) \tag{3}$$

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Show that the stereographic projection is only invertible on $S^2 \backslash \{(1,0,0)\}$ and determine its inverse. How many other coordinate patches are needed to cover S^2 ? Give these other patches and also the coordinate transformations that relates two patches.

Calculate the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ corresponding to the metric above. (Hint: Use the trick that the Euler Lagrange equations corresponding to the variation of $x^\mu(t)$ coming from the Lagrangian: $\frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}$ is the geodetic eqution from where it is easy to read off the Christoffel symbols.) If you used the hint, prove its statement.

Show that the curves $r(u) = \tan(u/2)$ and $\phi = \Phi$ (constant) are geodesics.

Calculate the transformation of an arbitrary vector that gets paralell transported along the curve: $r = R$ (constant), $\phi_i = \Phi$, $\phi_f = \Phi + \alpha$.

For which value of R the above curve is geodesic ?

Calculate the surface of the region $0 < r < R$.

21. Consider a smooth nonzero real function (scalar field) Ω on a manifold M with the metric $g_{\mu\nu}$. The metric $\Omega^2 g_{\mu\nu}$ is said to arise from $g_{\mu\nu}$ via a *conformal transformation*.

Write down the tensor $C^{\alpha}_{\mu\nu}$ which may be defined by the equation

$$\bar{\nabla}_\mu v_\nu = \nabla_\mu v_\nu - C^{\alpha}_{\mu\nu} v_\alpha \tag{4}$$

where $\bar{\nabla}_\mu$ is the Levi-Civita covariant derivative corresponding to $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

Show (or argue) that a metric is a positive multiple of another one at a point x if and only if the casual structure is the same for both at the point x. That is the notions spacelike, timelike and null concerning the norm of a vector at point x are invariant under conformal transformations and conversely: if they are invariant at one point then the metric is necessarily a positive multiple of the original metric at that point.

Show that the equation

$$v^\nu \nabla_\nu v^\mu = \alpha v^\mu \tag{5}$$

is the geodesic equations for the tangent $v^\mu = \frac{dx^\mu}{d\lambda}$ of a curve. I.e. it can be reparametrized so that the right hand side disappears.

Show that null geodesics are invariant under conformal transformations.

If a field $\bar{\Psi} = \Omega^s \Psi$ is a solution of a field equation corresponding to the metric $\Omega^2 g_{\mu\nu}$ as long as Ψ solves the field equation corresponding to $g_{\mu\nu}$ then it is called *conformally invariant*. Here, s is the conformal dimension of the field Ψ.

Show that the generalized Klein Gordon equation (for curved space) $g^{\mu\nu}\nabla_\mu \nabla_\nu \Phi = 0$ is conformally invariant only in two dimensions.

Show that the source free, curved space Maxwell equations ($\partial \to \nabla$) are conformally invariant in any dimensions.

Show that the conservation law $\nabla_\mu T^{\mu\nu} = 0$ for the stress-energy tensor of a physical system is conformally invariant only if $T^{\mu\nu}$ is symmetric and traceless.

Calculate the values of s in all three cases.