Introduction to General Relativity
exercises

22. Consider Einstein's equations in the presence of the cosmological term (7.34):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}$$

where we put back the velocity of light.
What is the dimension of Λ ?
Calculate the gravitational potential Φ in the Newtonian limit in the presence of the cosmological term.
Newton's potential,

$$\Phi = -G\frac{M_\odot}{r}$$

where $M_\odot = 2 \times 10^{30}$ kg is the mass of the Sun, gives a good description of gravitational forces within our planetary system.Use this fact to determine an upper limit for $|\Lambda|$, given the radius of Pluto's orbit $r_0 = 6 \times 10^{12}m$.
We can interpret the term

$$\frac{\Lambda c^4}{8\pi G}g_{\mu\nu} \tag{1}$$

as the stress-energy tensor of "empty space". The mass density of the universe $\rho_0 = 10^{-27}kg/m^3$. Compare this with the contribution of term (1) if we take for Λ the upper limit to be found above. Can you conclude from this comparison that the cosmological term plays an important role in cosmology ?

23. Do the indicated exercise in Chapter 8, that is, derive Eq. (8.6)
(Note: in the newest version of the lecture notes this is Eq. (8.7)).

24. Show the following identity used during the lecture for calculating the variation of the action in terms of the metric tensor.

$$\delta(\sqrt{-\det(g)}) = \frac{1}{2}\sqrt{-\det(g)}g_{\mu\nu}\delta g^{\mu\nu}$$

where $\delta g_{\mu\nu}$ is defined as in Eq. (8.5). (Hint: Prove the identity $\exp(\mathrm{tr}(A)) = \det(\exp(A))$ which holds for an arbitrary matrix. (Hint of Hint: Consider A in upper diagonal form.))