Introduction to General Relativity
exercises

25. Consider the physical system of a scalar field in flat space with a generic action:

$$S = \int d^4x \mathcal{L}(\Phi,\ \partial_\mu \Phi,\ x^\nu)$$

Show that under the transformation $x^\mu \rightarrow x^\mu + \delta x^\mu$, $\Phi \rightarrow \Phi + \delta\Phi$ the variation of the action will take the following form:

$$\int d^4x \left(\partial_\mu F^\mu + (e.o.m)\, \delta\Phi\right)$$

with

$$F^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi}\delta\Phi + \mathcal{L}\delta x^\mu$$

where $(e.o.m)$ means the equations of motions of the scalar field arising from the variation of the action with respect to the field Φ :

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \Phi} - \frac{\partial \mathcal{L}}{\partial \Phi} = 0$$

Thus we have $\partial_\mu F^\mu = 0$ in case of the above transformations were a symmetry of the system ($\delta S = 0$). F_μ is called the Noether current, $\int F^0 d^3x$ is called the Noether charge which is conserved. What is the Noether current for a displacement ? ($\delta x^\mu = a^\mu$, $\delta\Phi = \partial_\mu \Phi\, a^\mu$) What are the charges ? Show that substituting covariant derivatives in the place of ordinary derivatives in the expression for the Noether current we obtain the Hilbert definition for the stress-energy tensor (coming from the variation of the curved-space action with respect to the metric $g_{\mu\nu}$ (8.27):

$$T_{\mu\nu} = -D_\mu \Phi D_\nu \Phi + \frac{1}{2}\left((D_\alpha \Phi)^2 + m^2\Phi^2\right) g_{\mu\nu}$$

26. Do the exercises noted in chapter 10, which amount to proving that:

- The Bianchi identity (10.11) does not change if we make the substitution $\partial_\mu \rightarrow D_\mu$. Explain this.
- The curved space first Maxwell equation (10.12) can be rewritten as (10.13).
- (10.14) holds for the electromagnetic current.