Introduction to General Relativity
exercises

27. Besides learning how to find the Einstein tensor for geometries under given circumstances, one needs to be able to determine the stress-energy tensor of various matter distributions.
Calculate the stress-energy tensor in an inertial frame S of the following systems. Here, the gravitational fields generated by these sources are *not* yet considered, so we take space-time itself as yet to be flat: $g_{\mu\nu} = \eta_{\mu\nu}$.

    (a) A group of particles all moving with the same velocity $v^i = (v, 0, 0)$ as seen in S. Let the rest mass density of these particles be ρ_0 as measured in the comoving frame. Assume a high density of particles and treat them in the continuum approximation.
    (Hint: In the comoving frame of the particles there are neither energy or momentum current nor stresses ($T^{00} = -\rho_0$ is the only non-vanishing component). Thus one only needs to perform a Lorentz transformation and determine the transformed stress-energy tensor).

    (b) A ring of N similar particles of mass m rotating counter-clockwise in the $x - y$ plane about some point fixed in S at a radius a and angular velocity ω (The width of the ring is much less than a). Do not include the stress energy of whatever force keep them in orbit. Assume N is large enough that one can treat the particles as being continuously distributed.
    (Hint: for eg. $(x, y) = (0, a)$ we can import the solution from (a); and since all points on the ring are equivalent the polar $T^{\phi\mu}$ components will look exactly like $T^{x\mu}$ coordinates in (a) for $\mu = \phi$, t (x, t) respectively. Then ρ_0 should be expressed in terms of N and transformed back into Cartesian coordinates.

$$T^{\mu\nu} = -\frac{\gamma N \delta(\sqrt{x^2+y^2} - a)\delta(z)}{2\pi a} \times$$
$$\times \begin{pmatrix} 1 & -v\sin\phi & v\cos\phi & 0 \\ -v\sin\phi & v^2\sin^2\phi & -v^2\sin\phi\cos\phi & 0 \\ v\cos\phi & -v^2\sin\phi\cos\phi & v^2\cos^2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

    with $\gamma = \sqrt{1 - v^2}$ and $v = \omega a$).

    (c) Two such rings of particles, one rotating clockwise, the other counter-clockwise, at the same radius a. The particles do not collide or interact with each other in any way.
    (Hint: For the clockwise moving ring only the sign of v is reversed with respect to (b)).

    (d) Note that in the last two cases, $\partial_\mu T_{\mu\nu} \neq 0$, so that, before attempting to compute the associated gravitational field, one has to consider whatever force there is that keeps these object in orbit. Why is this so?

28. What is the stress-energy of a gas with proper number density (i.e. measured in the local rest frame of the gas) N of noninteracting particles of mass m, if all particles have the same speed v, but move isotropically ? (Do *not* assume $v \ll c$.) Specify the result for cold dust ($v = 0$) and for a homogeneous, incoherent light ray, or 'photon gas' ($v = c$ and $m\gamma = h\nu$) (Hint: $u^\mu = \gamma(1, \vec{n}v)$ where $\vec{n}$ is unit vector and $T^{\mu\nu} = -\kappa < u^\mu u^\nu >$ with κ is a constant related to the mass of the particles ($T^{00} = -m\gamma N$) and $<>$ denotes averaging over the orientations of $\vec{n}$.)

29. In the rest frame of a perfect fluid its strees-energy tensor, in terms of mass-density ρ and pressure p, is the diagonal tensor: $diag(-\rho, -p, -p, -p)$. If a fluid element of proper density ρ and pressure p is moving with a 4-velocity u^μ, what is it stress-energy ?
(Hint: express the rest frame expression in terms of covariant quantities and argue that it will thus have the same form in all frames).