Introduction to General Relativity
exercises

30. An observer $O_1$ sits in a rocket which moves in a circular orbit in the Schwarzschild universe. The orbit is characterized by

$$r > 2M, \quad \theta = \frac{\pi}{2}, \quad \frac{d\phi}{dt} = \omega \neq 0$$

where $\omega$ is the angular velocity. The nonzero connections are given by

$$\begin{aligned}
\Gamma^r_{tt} &= -\frac{M}{r^2}(1 - \frac{2M}{r}) \\
\Gamma^r_{rr} &= \frac{M}{r^2}(1 - \frac{2M}{r})^{-1} \\
\Gamma^r_{\theta\theta} &= r(1 - \frac{2M}{r}) \\
\Gamma^r_{\phi\phi} &= r(1 - \frac{2M}{r})\sin^2\theta
\end{aligned}$$

(a) For which values of $R$ does observer $O_1$ not need to use his rocket to stay in orbit ?

(b) What are the possible values of $\omega$ if the rocket motor can be used ?

(c) For which values of $R$ is the upper limit for $|\omega|$ maximal and what is the upper limit in that case ?

(d) Observer $O_2$ finds himself at

$$r = R, \quad \theta = \frac{\pi}{2}, \quad \phi = \phi_2$$

He transmits radiation with wavelength $\lambda$. This radiation is recieved by observer $O_3$ at

$$r = L > R, \quad \theta = \frac{\pi}{2}, \quad \phi = \phi_3$$

with wavelength $\lambda'$. Calculate the redshift $\frac{\lambda'}{\lambda}$.