Introduction to General Relativity
exercises

31. In the Schwarzschild geometry a test particle of relativistic velocity v flies past a mass M at an impact parameter r_0 so great that the deflection angle Δ_{grav} is small. Calculate Δ_{grav}. (Hint: follow the lines of the 12th chapter of the lecture notes for the calculation of the bending of light rays. Note however, that the world-line of the test particle is now *timelike*. Show that the result coincide with (12.38) when $v = c$).
In flat space a test charge e flies with velocity v past a nucleus of charge Ze with impact parameter[1] r_0 so great that the deflection angle Δ_{em} is small. Calculate Δ_{em}. Why is the formula for Δ_{grav} different from that for Δ_{em} ?
Also do the exercise indicated at the end the chapter 12 of the lecture notes: calculate the "bending" of the trajectory of a nonrelativistic particle in a hyperbolic orbit around a star.

32. Show that (12.27) $\delta\phi = 2\pi\frac{3M^2}{J^2}$ for the perihelion shift in ordinary units reads as
$$\delta\phi = \frac{6\pi G m_\odot}{a(1-\varepsilon^2)c^2}$$
with $m_\odot$ being the mass of the central star, and ε is the excentricity of the orbit.

33. Show that once a rocket ship crosses the horizon $r = 2M$ in the Schwarzschild geometry, it will reach $r = 0$ in a proper time $\tau \leq \pi M$ no matter how the engines are fired. (Hint: $g_{\mu\nu}u^\mu u^\nu = -1$ for a timelike observer and use that $u^r < 0$ within the horizon after convincing yourself about its validity.

34. Find the non-rotating magnetic monopole solution by postulating a radial magnetic field.

35. Consider the metric
$$\mathrm{d}s^2 = -\mathrm{d}t^2+\mathrm{d}z^2+L^2(e^{2\beta}\mathrm{d}x^2+e^{-2\beta}\mathrm{d}y^2) = -\mathrm{d}u\mathrm{d}v+L^2(e^{2\beta}\mathrm{d}x^2+e^{-2\beta}\mathrm{d}y^2)$$
with $u = t + z$, $v = t - z$. and $L = L(u)$, $\beta = \beta(u)$. This means that L and β *o*nly depend on the cordinates *via* the variable u.

    (a) Calculate the Christoffel symbols corresponding to the above metric in (u, v, x, y) coordinates.

    (b) Show that the curves
    $$\dot{t} = -1, \quad \dot{x} = \dot{y} = \dot{z} = 0$$
    are timelike geodesics.

[1] That is the smallest distance from the central mass.

(c) Demonstrate that the only non-zero component of the Ricci tensor is given by

$$R_{uu} = -2\left(\frac{L''}{L} + \beta'^2\right).$$

Assuming the absence of matter, we conclude that for given $\beta(u)$ and initial conditions $L(u_0)$, $L'(u_0)$, with the restriction that $L(u_0) \neq 0$, one can calculate $L(u)$ from the Einstein equations. This gives an exact plane gravitational wave.

(d) Take $\beta(u) = 0$ and $L(u) = 1$ for $u > 0$ and $\beta(u) = u$ for $u \geq 0$. Show that L reaches for finite u the value zero if $\beta \neq 0$ (β continuous differentiable). What does this imply for particles which originally were at rest in the $z = z_0$ plane ?

(e) Show that particles which originally were at rest with the same x and y coordinates keep the same physical distance to each other. We deduce that the wave is transversal.

(f) Show that this space-time is flat in the regions were $\beta(u) = 0$.