Introduction to General Relativity
exercises

5. Form “natural” units of mass, length and time out of the physical constants $\hbar$, G and c. Calculate their values.

6. Estimate the Bohr radius of a “gravitational” atom, e.g. two neutrons bound by their gravitational attraction in their lowest energy state.

7. Consider the star operation on a two-form in four dimensional Minkowski space given by

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{1}$$

   Show that, when it is applied on the electromagnetic field strength tensor, then $E \to B$ and $B \to -E$ for the three-vectors of the electric and magnetic field.

   Show that if $F_{\mu\nu}$ is a solution of the free-space Maxwell equations, so is $\star F$ and also $e^{\star a}F \equiv F\cos\alpha + \star F\sin\alpha$ for arbitrary α. (Note that the latter transformation is called “duality rotation”).

8. As it was explained during the lectures, the notion of symmetry (and antisymmetry) is an invariant property of tensors with an arbitrary number of indices. Consider the useful and widely used definitions for the completely symmetrized (and antisymmetrized) part of an arbitrary tensor:

$$T_{(\alpha_1\alpha_2\ldots\alpha_n)} = \frac{1}{p!}\sum_P T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{2}$$

$$T_{[\alpha_1\alpha_2\ldots\alpha_n]} = \frac{1}{p!}\sum_P (-1)^{|P|}T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{3}$$

   Here the summation is for all permutation P of the indices, $(-1)^{|P|}$ is the parity, -1 for odd, 1 for even permutations.

   How many independent components do these tensors have (for arbitrary n) ?

   How many independent components do these tensors have if the dimension d is also arbitrary?

   Use the result to show that for the generic case (that is when the tensor is not constrained by any symmetry property) the tensor $T_{\alpha_1\alpha_2\ldots\alpha_n}$ itself is **not** the sum of the completely symmetric and antisymmetric part for $n > 2$.

Of course the process for tensors with purely upper indices the procedure is identical. Can you symmetrize in mixed indices ? That is, e.g. is $T^\mu_\nu + T^\nu_\mu$ a tensor ?

9. You are given a tensor $K^{\alpha\beta}$. How can you test whether it's a direct product of two vectors $K^{\alpha\beta} = A^\alpha B^\beta$ or not ? Is your test independent of the choice of coordinates ?

10. Prove (5.5) in the lecture notes: show that two successive transformations of that form produce one of the same form.

11. Check (5.8a).

12. Derive the expression (5.27) for the curvature starting from Eq. (5.31) as a *definition* of the Riemann tensor.

13. Suppose, that we define the covariant derivative abstractly as the linear map, which obeys Leibnitz rule: (5.15), commutes with contraction: (5.16), and acts as ordinary derivative on scalar fields (5.17). Show (or argue) that the difference of two derivative operators gives rise to a tensor C^c_{ab}. (That is $(\nabla_\mu - \tilde{\nabla}_\mu)A_\nu = C^\alpha_{\mu\nu}A_\alpha$. (Hint: The difference at point P depends only on the value of the vector field A_μ at that point and it is also a linear transformation.) Show that it is symmetric in its lower indices, if we require that $\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu$ vanishes on scalar fields. (In the general case its antisymmetric part is called torsion.) Show, using (5.9) for the covariant derivative, that this is equivalent of the symmetricity of the quantity $\Gamma^\alpha_{\mu\nu}$

14. Derive the Bianchi identity (5.30) in the coordinate frame where

$$\Gamma^\mu_{\nu\alpha} = 0 \ , \tag{4}$$

following the hints in the text after (5.30).

15. Prove the following symmetry properties (5.26) and (5.29)of the Riemann curvature tensor, supposing (5.27) to be the definition. Argue that, for the latter, we may also use the special frame where $\Gamma^\mu_{\nu\alpha}$ vanishes locally.

16. Using the symmetry properties given in Chapter 5, derive the number of *mutually independent* components of the Riemann tensor.

    When we will know how to raise and lower indices we will find one more symmetry property of the Riemann tensor:

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu} \tag{5}$$

    How many independent components does the Riemann tensor have then? How many does it have in d dimensions?

17. Consider a manifold M with metric $g_{\mu\nu}$. Take a point $x \in M$ fixed. Show that there exists a special coordinate system in which the connections coefficients are zere at that point.

$$\Gamma^{\alpha}_{\mu\nu}(x) = 0 \tag{6}$$

    Such a coordinate system is called a *geodesic coordinate system*. Note: if the above property holds for the whole M in one and the same coordinate system, then the manifold is called *affine flat*. Show that we may furthermore choose our coordinates such that the metric is of the form:

$$g_{\mu\nu}(x) = \mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1). \tag{7}$$

    Show that (1) (for all values of the indices α, μ, ν) implies

$$\partial_\alpha g_{\mu\nu}(x) = 0. \tag{8}$$

    Property (1), (2) and (3) defines the notion of a *local inertial frame* at x. Which transformations preserve the properties of a local inertial frame ?

    Choose a local inertial frame about x. Let the coordinates of x be the origin of the frame. Show that in a neighbourhood of x the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}(\partial_\alpha \partial_\beta g_{\mu\nu}) x^\alpha x^\beta \tag{9}$$

    Compute the Riemann curvature, the Ricci tensor and the Ricci scalar at the point x. For simplicity, assume $\eta_{\mu\nu} = \delta_{\mu\nu}$.

18. A space purports to be 3-dimensional with coordinates x, y, z and the metric

$$ds^2 = dx^2 + dy^2 + dz^2 - (\frac{3}{13}dx + \frac{4}{13}dy + \frac{12}{13}dz)^2. \tag{10}$$

Show that it is really a two-dimensional space and find two new coordinates χ, η for which the line element takes the form

$$ds^2 = d\chi^2 + d\eta^2 \tag{11}$$

. Hint: Calculate the volume element or density function for this space to convince yourself, that the coordinates are linearly dependent so that we can omit one and perform a linear (orthogonal?) transformation to obtain the 2-dimensional Euclidean line element (6).

19. Show that a contraction of a vector with the projection tensor: $g_{\mu\nu} + \epsilon u_\mu u_\nu$ projects the vector into the 3-surface orthogonal to the vector u_μ, where $\epsilon = 1$ if u_μ is timelike, $\epsilon = -1$ if u_μ is spacelike, and $|u^\mu u_\mu| = 1$ for these cases. Show that there is no unique projection orthogonal to a lightlike or null vector.