Introduction to General Relativity
exercises

5. Form "natural" units of mass, length and time out of the physical constants $\hbar$, G and c. Calculate their values.

6. Estimate the Bohr radius of a "gravitational" atom, e.g. two neutrons bound by their gravitational attraction in their lowest energy state.

7. Consider the star operation on a two-form in four dimensional Minkowski space given by

$$\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{1}$$

Show that, when it is applied on the electromagnetic field strength tensor, then $E \to B$ and $B \to -E$ for the three-vectors of the electric and magnetic field.

Show that if $F_{\mu\nu}$ is a solution of the free-space Maxwell equations, so is $\star F$ and also $e^{\star a}F \equiv F\cos\alpha + \star F \sin\alpha$ for arbitrary α. (Note that the latter transformation is called "duality rotation").

8. As it was explained during the lectures, the notion of symmetry (and antisymmetry) is an invariant property of tensors with an arbitrary number of indices. Consider the useful and widely used definitions for the completely symmetrized (and antisymmetrized) part of an arbitrary tensor:

$$T_{(\alpha_1\alpha_2\ldots\alpha_n)} = \frac{1}{p!}\sum_P T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{2}$$

$$T_{[\alpha_1\alpha_2\ldots\alpha_n]} = \frac{1}{p!}\sum_P (-1)^{|P|} T_{\alpha_{P(1)}\alpha_{P(2)}\ldots\alpha_{P(n)}} \tag{3}$$

Here the summation is for all permutation P of the indices, $(-1)^{|P|}$ is the parity, -1 for odd, 1 for even permutations.

How many independent components do these tensors have (for arbitrary n) ?

How many independent components do these tensors have if the dimension d is also arbitrary?

Use the result to show that for the generic case (that is when the tensor is not constrained by any symmetry property) the tensor $T_{\alpha_1\alpha_2\ldots\alpha_n}$ itself

is **not** the sum of the completely symmetric and antisymmetric part for $n > 2$.

Of course the process for tensors with purely upper indices the procedure is identical. Can you symmetrize in mixed indices ? That is, e.g. is $T^{\mu}_{\nu} + T^{\nu}_{\mu}$ a tensor ?

9. You are given a tensor $K^{\alpha\beta}$. How can you test whether it's a direct product of two vectors $K^{\alpha\beta} = A^{\alpha}B^{\beta}$ or not ? Is your test independent of the choice of coordinates ?