1. Let's take the divergence of (1.24).

$$\partial_\mu T^{em}_{\mu\nu} = (\partial_\mu F_{\mu\lambda})F_{\lambda\nu} + F_{\mu\lambda}\partial_\mu F_{\lambda\nu} + \frac{1}{2}\delta_{\mu\nu}F_{\lambda\sigma}\partial_\mu F_{\lambda\sigma} =$$
$$= -J_\lambda F_{\lambda\nu} + F_{\mu\lambda}(\frac{1}{2}\partial_\mu F_{\lambda\nu} + \frac{1}{2}\partial_\lambda F_{\nu\mu} + \frac{1}{2}\partial_\nu F_{\mu\lambda}) = F_{\nu\lambda}J_\lambda \tag{1}$$

For the first term we used the first Maxwell equation (1.16), the second has been split up in the following way:

$$F_{\mu\lambda}\partial_\mu F_{\lambda\nu} = \frac{1}{2}F_{\mu\lambda}\partial_\mu F_{\lambda\nu} + \frac{1}{2}F_{\lambda\mu}\partial_\lambda F_{\mu\nu} = \frac{1}{2}F_{\mu\lambda}\partial_\mu F_{\lambda\nu} + \frac{1}{2}F_{\mu\lambda}\partial_\lambda F_{\nu\mu}$$

that is one half of the expression left untouched, in the other half summation indices μ and λ have been interchanged and the last equation is just simultaneous interchange of the indices of both F 's resulting in the factor of $(-1)^2$ by the antisymmetry property. In the third term summation for μ has been performed obviously then summations for λ and σ have been changed to μ and λ respectively. Finally the expression in brackets equals to zero by the second Maxwell equation (1.17) known as the Bianchi identity and the antisymmetricity of $F_{\mu\lambda}$ was used again to get the right hand side of (1).
For interpreting (1) let's integrate it over a spatial hypersurface V separating the time and the spatial components. For $\nu = 0$ we have (leaving the superscript "em"):

$$\partial_0 \int_V T_{00} d^3x = -\int_V \partial_i T_{i0} d^3x + \int_V F_{0i}J_i d^3x = -\int_{\delta V} T_{i0} dS + \int_V F_{0i}J_i d^3x$$

T_{00} is the energy density, thus the change of the total energy of the system comes from the energy current density T_{i0} at the boundary of V and the part of the (electromegnetic) current which is paralell to the electric field F_{0i}. If V is infinite with boundary only at infinity with the natural assumption of $T_{i0}\mid_\infty = 0$ we have only the explained second term of the right hand side. For $\nu = i$ the equation takes the form:

$$\partial_0 \int_V T_{0i} d^3x = -\int_V \partial_j T_{ji} d^3x + \int_V F_{i\lambda}J_\lambda d^3x =$$
$$= -\int_{\delta V} T_{ji} dS + \int_V F_{i0}J_0 d^3x + \int_V F_{ij}J_j d^3x$$

The explanation is analogous. T_{0i} is the momentum density T_{ji} is the momentum current density (known as stress tensor), and the momentum increases with $F_{i0}J_0 = \rho E_i$ and $F_{ij}J_j = (\overrightarrow{J} \times \overrightarrow{B})_i$.

2. Instead of the matrix $A_{ij} \equiv \frac{\partial x^i}{\partial x'^j}$ we write down the inverse immediately:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$

We save time with the following trick

$$\begin{aligned}
\triangle = \frac{\partial^2}{\partial x^{i2}} = (A^{-1})_{ij}\frac{\partial}{\partial x'^j}(A^{-1})_{ik}\frac{\partial}{\partial x'^k} = \\
= (A^{-1})_{ij}\frac{\partial (A^{-1})_{ik}}{\partial x'^j}\frac{\partial}{\partial x'^k} + (AA^T)^{-1}_{jk}\frac{\partial^2}{\partial x'^j \partial x'^k} = \\
= \left(\frac{\partial \left(AA^T\right)^{-1}_{jk}}{\partial x'^j} - \frac{\partial A^{-1}_{ij}}{\partial x'^j} A^{-1}_{ik} \right) \frac{\partial}{\partial x'^k} + (AA^T)^{-1}_{jk}\frac{\partial^2}{\partial x'^j \partial x'^k} \\
= \frac{2}{r}\frac{\partial}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial r^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} =
\end{aligned}$$

where AA^T will be diagonal and only the second term in the big parenthesis requires considerable amount of work. So this gives the convenient form for the Schrodinger equation so as to calculate the spectrum.

$$(-\frac{\hbar^2}{2m}\triangle_{polar} - \frac{e^2}{r})\psi(r,\theta,\phi) = \psi(r,\theta,\phi)$$

3. The differential equation is a trivial step from formula (3.8) in the lecture notes.

$$L(\tau + d\tau) = L(\tau) + \dot{L}(\tau)d\tau = L(\tau) + \delta L\, L(\tau)d\tau$$

That is

$$\dot{L}(\tau) = \delta L\, L(\tau)$$

with

$$\delta L = \begin{pmatrix} 0 & 0 & 0 & -ig_1(\tau) \\ 0 & 0 & 0 & -ig_2(\tau) \\ 0 & 0 & 0 & -ig_2(\tau) \\ ig_1(\tau) & ig_2(\tau) & ig_3(\tau) & 0 \end{pmatrix} d\tau$$

Integration after separating the variables yields:

$$L(\tau) = \exp(\delta L)$$

which is easily summed for $\vec{g}(\tau)$ pointing in the z-direction it gives formula (3.9).

4. From the reference frame of the Earth the proper time of the cannonball to the first non-trivial order in $\frac{v^2}{c^2}$ is:

$$\begin{aligned}
\tau = \int_0^{2v_0/g} \sqrt{1-\frac{v^2}{c^2}}dt \simeq \int_0^{2v_0/g}(1-\frac{v^2}{2c^2})dt = \int_0^{2v_0/g}(1-\frac{(v_0-gt)^2}{2c^2})dt = \quad (2) \\
= \int_{-v_0/g}^{v_0/g}(1-\frac{g^2t'^2}{2c^2})dt' = \frac{2v_0}{g}(1-\frac{v_0^2}{6c^2}) = t(1-\frac{v_0^2}{6c^2})
\end{aligned}$$

Repeating the calculation for the proper time of the Earth from the moving grid of the cannonball is identical to the one above if we ignore possible gravitational effects:

$$t = \tau(1-\frac{v_0^2}{6c^2})$$

expressing τ neglecting higher orders of v_0^2/c^2 we get:

$$\tau = t(1 + \frac{v_0^2}{6c^2}) \tag{3}$$

which leads to the twin paradox when comparing it to (2).In fact the last result is correct since the freefalling cannonball does not feel any gravitation. However the first calculation has to be modified by taking into account the effect of the gravitational potential when writing down the the clock speed of the cannonball as noted in the exercise Pluging back c^2 to the formulae the calculation is as follows:

$$\begin{aligned}\tau = \int_0^{2v_0/g} (\sqrt{1 - \frac{v^2}{c^2}} + \frac{g}{c^2} z) dt \simeq \frac{2v_0}{g}(1 - \frac{v_0^2}{6c^2}) + \int_0^{2v_0/g} \frac{g}{c^2} z dt = \\ \frac{2v_0}{g}(-\frac{v_0^2}{6c^2}) + \int_0^{2v_0/g} \frac{g}{c^2}(v_0 t - \frac{g}{2}t^2) dt \simeq \frac{2v_0}{g}(1 - \frac{v_0^2}{6c^2} + \frac{v_0^2}{3c^2}) = \\ = t(1 + \frac{v_0}{6c^2})\end{aligned}$$

which agrees with (3) and shows the somewhat unexpected result, that is the freefalling observer in fact older than the one feeling gravitational force. The usual formulation of the twin paradox leads to the opposite answer, when one takes his racket accelerates to a high velocity and comes back to the Earth, one is in deed yonger than his twin brother on the Earth, however he was not freely falling along his journey, while the cannonball was.

5.

$$L_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} m$$

$$T_P = \sqrt{\frac{\hbar G}{c^5}} = 5.391 \times 10^{-44} s$$

$$M_P = \sqrt{\frac{\hbar c}{G}} = 2.177 \times 10^{-8} kg$$

6. Force balance gives us

$$\frac{Gm_N}{r^2} = m_N \omega^2 \left(\frac{1}{2} r\right)$$

According to Bohr quantization the separation radius must satisfy:

$$2m_N \omega \left(\frac{1}{2} r\right)^2 = n\hbar$$

From these two equations, the separation is:

$$r = \frac{2n^2\hbar^2}{Gm_N}$$

The lowest energy level is the $n = 1$ level, in this level the radius is $6 \times 10^{22} m = 6 \times 10^6$ light years!

7. We know that

$$E_i = iF_{i4}$$
$$B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$$

For $\star F$ these become

$$i\star F_{i4} = i\frac{1}{2}\varepsilon_{i4jk}F_{jk} = \frac{1}{2}\varepsilon_{ijk}F_{jk} = iB_i$$

$$\begin{aligned}\frac{1}{2}\varepsilon_{ijk}\star F_{jk} = \frac{1}{2}\varepsilon_{ijk}\frac{1}{2}\varepsilon_{jk\mu\nu}F_{\mu\nu} = \frac{1}{4}\varepsilon_{ijk}(\varepsilon_{jk4l}F_{4l} + \varepsilon_{jkl4}F_{l4}) = \\ = \frac{1}{4}2\varepsilon_{ijk}\varepsilon_{jkl}F_{l4} = \frac{1}{4}2\cdot 2\delta_{il}F_{l4} = F_{i4} = -iE_i\end{aligned}$$

The latin letters run from 1..3, the antisymmetricity of F has been used and the identity $\varepsilon_{ijk}\varepsilon_{jkl} = 2\delta_{il}$. Note that the factor of i entered because of the form (1.13) of $F_{\mu\nu}$ compatible with the purely imaginary time components of vectors.
The Bianchi identity (1.17):

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{4}$$

is equivalent with

$$\partial_\mu \star F_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = 0 \tag{5}$$

Clearly, for fixed ν and different $(\mu,\, \alpha,\, \beta,\, \nu)$ in (5), all the combinations of $(\mu,\, \alpha,\, \beta)$ appears in the sum for the ∂F term twice; the odd permutations with the same sign and the even ones with the opposite sign. Again by the antysimmetricity of F, this gives $\pm$twice (4). For different $(\alpha,\, \beta,\, \gamma)$ in (4) one term is zero and the other two are equal with different sign again by antisymmetry. So we can write the free space Maxwell's equations in the form:

$$\partial_\mu F_{\mu\nu} = 0 \tag{6}$$
$$\partial_\mu \star F_{\mu\nu} = 0 \tag{7}$$

We need the property of the operator $\star$:

$$\star\star F_{\mu\nu} = F_{\mu\nu}$$

which follows from the first part ($E_i \to iB_i \to i(-iE_i) = E_i$, similarly for B_i). Thus "staring" F in (6) and (7) interchanges the equations. Due to linearity we can also take arbitrary linear combination of F and $\star F$, it will also be solution.

8. In d dimensions an n index antisymmetric tensor has as many components as many different n-tuple we can form of d different numbers with the restriction that they all have to be different. The solution is $\begin{pmatrix} d \\ n \end{pmatrix} =$

$\frac{d(d-1)\cdot\ldots\cdot(d-n+1)}{n!}$. For the symmetric case the equivalent combinatorical question is: How many different ways can we distribute less or precisely n balls into $d-1$ different boxes. The reason is that one independent component of a fully symmetric n tensor in d dimensions is characterized by the numbers of "1"s, "2"s,...,"(d-1)"s appearing among the indices regardless their distribution among the indices ("d"s are fixed since the rest of the indices should take the value of "d". The solution is $\begin{pmatrix} d+n-1 \\ n \end{pmatrix} = \frac{(d+n-1)(d+n-2)\cdot\ldots\cdot d}{n!}$ since there are $(d+n-1)!$ permutations of the boxes and balls together to be devided by the irrelevant permutations of the boxes by themselves $(d-1)!$ and the balls $n!$.
In one hand writing down the transformation of an object of the type $T_\mu^\nu \pm T_\nu^\mu$ gives a quantity which is not symmetric or antisymmetric in general, on the other hand identifying a superscript index with a subscript index makes no sense (they do not represent any intrinsic coordinate independent quantity), they live in different space.

9.

$$K^{\alpha\beta}v_\beta = A^\alpha B^\beta v_\beta = (B\cdot v)A^\alpha$$

The equation is a covariant one, it shows that contracting our tensor with any covector gives a contravector pointing to the same direction (A^α). So it is a necessary condition. It is also sufficient, since picking up $\widehat{A}^\alpha \equiv K^{\alpha\beta}v_\beta^{(0)}$ for a fixed $v_\beta^{(0)}$ (which does not give identically zero $\widehat{A}^\alpha$) we can define B^β by its contractions with arbitrary covectors $B^\beta v_\beta \equiv \frac{K^{\alpha\beta}v_\beta}{\widehat{A}^\alpha}$which makes sense as the vector in the numerator is proportional to the vector in the denominator.

10. In a general coordinate system eqn. (5.3) holds Consider the transformation :$u^\mu \to w^\mu(u)$. We write down (5.3) for the coordinates w^μ while on the other hand $\dot{\xi}^\mu$ is a contravector, thus substituting (5.3) into the equation $\dot{\xi}^\mu(w) = w^\mu_{,\nu}\dot{\xi}^\nu(u)$ for both the coordinates u^μ and w^μ and plugging in the transformation rules for the quantities in (5.3) corresponding to the w coordinates we get:

$$\begin{aligned} w^\mu_{,\nu,\alpha}\dot{u}^\alpha\xi^\nu + w^\mu_{,\nu}\frac{d}{d\tau}\xi^\nu(u) + \Gamma^\mu_{\alpha\beta}(w)w^\beta_{,\delta}\dot{u}^\delta w^\alpha_{,\gamma}\xi^\gamma(u) = \\ w^\mu_{,\nu}\left(\frac{d}{d\tau}\xi^\nu(u) + \Gamma^\nu_{\alpha\beta}(u)\dot{u}^\beta\xi^\alpha(u)\right) \end{aligned}$$

The term containing $\frac{d}{d\tau}\xi^\nu(u)$ appears on both sides, $\dot{u}^\beta\xi^\alpha$ is contracted in all the rest of the terms, so renaming the indices appropriately and after rearrangement we get:

$$\dot{u}^\beta\xi^\alpha\left(w^\mu_{,\alpha,\beta} + \Gamma^\mu_{\delta\gamma}(w)w^\gamma_{,\beta}w^\delta_{,\alpha} - w^\mu_{,\nu}\Gamma^\nu_{\alpha\beta}(u)\right) = 0$$

for arbitrary curve $u(\tau)$ and arbitrary vector field ξ^μ along that curve, so the bracket must vanishes which is precisely the transformation (5.5) if multiplied by $u^\lambda_{,\mu}$:

$$u^\lambda_\mu\left(w^\mu_{,\alpha,\beta} + \Gamma^\mu_{\delta\gamma}(w)w^\gamma_{,\beta}w^\delta_{,\alpha}\right) = \Gamma^\lambda_{\alpha\beta}(u)$$

The second part of the exercise is straightforward, one has to consider two successive transformations $y \to x(y)$ and $x \to u(x)$ resulting in the transformation $y \to u(x(y))$. Plugging in the the formula for $\Gamma(x(y))$ in the place of $\Gamma(x)$ in (5.5) identifying the consequences of the chain rule for differentiation: $x^{\mu}_{,\alpha}(u) y^{\nu}_{,\mu}(x) = \frac{\partial y^{\nu}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial u^{\alpha}} = \frac{\partial y^{\nu}}{\partial u^{\alpha}} = y^{\nu}_{,\alpha}(u)$ and for identifying the second derivative:

$$y^{\mu}_{,\kappa,\lambda}(u) = y^{\mu}_{,\alpha,\beta}(x) x^{\alpha}_{,\lambda}(u) x^{\beta}_{,\kappa}(u) + y^{\mu}_{,\alpha}(x) x^{\alpha}_{,\kappa,\lambda}(u)$$

which comes from applying the chain rule twice, (5.5) is obtained for $y \to u(y)$.

11. Reparametrizing eqn. (5.8) using the chain rule twice:

$$\frac{dx^{\mu}(\sigma)}{d\sigma'} = \frac{dx^{\mu}(\sigma)}{d\sigma}\frac{d\sigma}{d\sigma'}$$

$$\frac{d^2x^{\mu}(\sigma)}{d\sigma'^2} = \frac{d^2x^{\mu}(\sigma)}{d\sigma^2}\left(\frac{d\sigma}{d\sigma'}\right)^2 + \frac{dx^{\mu}}{d\sigma}\frac{d^2\sigma}{d\sigma'^2}$$

we immediately get eqn. (5.8a) with $\alpha(\sigma') = \frac{d^2\sigma}{d\sigma'^2}$.

12. Applying the rules of the transformation of a covector (for getting $D_\nu A_\alpha$) then that of a two-index cotensor (for getting $D_\mu D_\nu A_\alpha$) we obtain the following:

$$D_\mu D_\nu A_\alpha = D_\mu(\partial_\nu A_\alpha - \Gamma^{\beta}_{\nu\alpha}A_\beta) = \partial_\mu\partial_\nu A_\alpha - \Gamma^{\beta}_{\mu\nu}\partial_\beta A_\alpha - \Gamma^{\beta}_{\mu\alpha}\partial_\nu A_\beta - \\ -\partial_\mu(\Gamma^{\beta}_{\nu\alpha}A_\beta) + \Gamma^{\gamma}_{\mu\nu}\Gamma^{\beta}_{\gamma\alpha}A_\beta + \Gamma^{\gamma}_{\mu\alpha}\Gamma^{\beta}_{\nu\gamma}A_\beta$$

The first, the second, the fifth member of the sum, the third plus the $\Gamma^{\beta}_{\nu\alpha}\partial_\mu A_\beta$ part of the fourth are symmetric with respect to the interchange of μ and ν (On the account of the Yang's theorem for partial derivatives, and torsionfree connection). Therefore after antisymmetrizing so as to write down (5.31) we get:

$$-R^{\beta}_{\alpha\mu\nu}A_\beta \equiv (-\partial_\mu\Gamma^{\beta}_{\nu\alpha} + \partial_\nu\Gamma^{\beta}_{\mu\alpha} + \Gamma^{\gamma}_{\mu\alpha}\Gamma^{\beta}_{\nu\gamma} - \Gamma^{\gamma}_{\nu\alpha}\Gamma^{\beta}_{\mu\gamma})A_\beta$$

which coincide with (5.27) for any torsionfree connection.

13. We need to show that $(D_\mu - \widetilde{D}_\mu)v_\nu|_P$ depends only on the value of v_ν at point P. This is true since $(D_\mu - \widetilde{D}_\mu)\phi v = \phi(D_\mu - \widetilde{D}_\mu)v$ due to the fact that the Leibniz rule and (5.17) hold for both covariant derivatives. (To be more elaborate one should take the $D_\mu - \widetilde{D}_\mu$ acting on $v_\nu - \bar{v}_\nu$ where the latter vanishes at P but different from zero in general and show that the result is zero everywhere.) Since one has the equation

$$D_\mu D_\nu \phi = \widetilde{D}_\mu \widetilde{D}_\nu \phi - C^{\alpha}_{\mu\nu}\partial_\alpha\phi$$

symmetricity in lower indices follows from the requirement of $D_\mu D_\nu - D_\mu D_\nu$ vanishing on scalar fields for both covariant derivatives. Using the explicit form (5.9) we find

$$0 = (D_\mu D_\nu - D_\mu D_\nu)\phi = \partial_\mu\partial_\nu\phi - \Gamma^{\alpha}_{\mu\nu}\partial_\alpha\phi - \partial_\nu\partial_\mu\phi + \Gamma^{\alpha}_{\nu\mu}\partial_\alpha\phi = \\ = (\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu})\partial_a\phi$$

that is $\Gamma^{\alpha}_{\mu\nu}$ has to be symmetric in its lower indices.

14. In the frame where $\Gamma^{\alpha}_{\mu\nu}|_P = 0$[1]

$$D_\beta R^\nu_{\kappa\lambda\alpha} = \partial_\beta\partial_\lambda\Gamma^\nu_{\kappa\alpha} - \partial_\beta\partial_\alpha\Gamma^\nu_{\kappa\lambda}$$

Considering the symmetrization in $(\beta,\, \lambda,\, \alpha)$ one easily checks that each $\partial\partial$ term will appear antisymmetrized in the complete sum and we get identically zero which asserts the proof of the Bianchi identity.

15. (5.26) is clear again in the coordinate system where $\Gamma^{\alpha}_{\mu\nu}|_P$ vanishes. (Note, that (anti)symmetricity is an invariant property). (5.29) in the same coordinate system is also straightforward as each index in the derivative appears twice and with different sign and the order of the other two indices does not matter since $\Gamma^{\alpha}_{\mu\nu}$ is symmetric in the lower indices.

16. We have antisymmetry in the first pair of indices and the last pair of indices of $R_{\alpha\beta\gamma\delta}$. There are thus $\frac{d(d-1)}{2}$ candidates for index pairs both for the first and the last pair. Additional symmetry property for exchanging the first pair with the last leads us to

$$\frac{1}{2}\left(\frac{d(d-1)}{2}\right)\left(\frac{d(d-1)}{d}+1\right) \tag{8}$$

. The more difficult part is to determine how many more components (5.29) makes dependent. From now on, different Greek letters mean different values. Writing the permutations out symbolicly and using the antisymmetric properties:

$(\alpha\beta\alpha\delta)$ (5.29): $0 = (\alpha\beta\alpha\delta)+(\alpha\delta\beta\alpha)+(\alpha\alpha\delta\beta) = (\alpha\beta\alpha\delta)-(\alpha\beta\alpha\delta)+0 \equiv 0$

$(\alpha\beta\alpha\beta)$ (5.29): $0 = (\alpha\beta\alpha\beta)+(\alpha\beta\beta\alpha)+(\alpha\alpha\beta\beta) = (\alpha\beta\alpha\beta)-(\alpha\beta\alpha\beta)+0 \equiv 0$

$(\alpha\beta\gamma\delta)$ We state that (5.29) is not sensitive for permuting the four indices, that is we have one equation for all four tuples regardless the order of the indices in it.

(a) Writing (5.29) if any of the last three indices are interchanged leads obviously to (5.29) for the original order as the odd permutations of the last three indices coincide with the evens on the account of antisymmetry.

(b) Interchanging any of the last three indices with the first gives -1 times (5.29) for the original order since say we interchanged μ and ν then we match $(\mu,\nu,\ ,\ )$ with $(\nu,\mu,\ ,\ )$; $(\mu,\rho,\ ,\ )$ with $(v,\kappa,\ ,\ )$ etc., and the reason we do not have to worry about not getting always the -1 times the matched tuple (according to the symmetry properties) is that in the interchanged equation we have only odd permutations of the indices.

[1] Note that the vanishing of the totally (anti)symmetric part of an arbitrary subset of indices (of the same kind) of a tensor is an invariant statement since the coordinate transformations leave (anti)symmetry and the matrix **0** invariant.

Hence we need to subtract the number of different four tuples $\begin{pmatrix} d \\ 4 \end{pmatrix}$ from (8), which leads to

$$\frac{1}{2}\left(\frac{d(d-1)}{2}\right)\left(\frac{d(d-1)}{d}+1\right)-\frac{d(d-1)(d-2)(d-3)}{4}=$$
$$\frac{d^2(d^2-1)}{12}=\begin{cases} 1 & d=2 \\ 6 & d=3 \\ 20 & d=4 \end{cases}$$

17. Consider (5.3) multiplied with $x^\gamma_{,\nu}$:

$$x^\gamma_{,\nu}\widetilde{\Gamma}^\nu_{\kappa\lambda}(u)=x^\alpha_{,\kappa}x^\beta_{,\lambda}\Gamma^\gamma_{\alpha\beta}(x)+x^\gamma_{\kappa,\lambda} \tag{9}$$

This form may suggest a guess for a coordinate transformation of the form (We want to make Γ vanish at point P which corresponds to the origin in coordinate u^μ):

$$x^\mu=u^\mu+\frac{1}{2}A^\mu_{\alpha\beta}u^\alpha u^\beta$$

with $A^\mu_{\alpha\beta}$ being symmetric in its lower indices. Plugging it into (9) one finds:

$$(\delta^\gamma_\nu+A^\gamma_{\alpha\nu}u^\alpha)\widetilde{\Gamma}^\nu_{\kappa\lambda}(u)=(\delta^\alpha_\kappa+A^\alpha_{\delta\kappa}u^\delta)(\delta^\beta_\lambda+A^\beta_{\delta\lambda}u^\delta)\Gamma^\gamma_{\alpha\beta}(x)+A^\gamma_{\kappa\lambda}$$

Setting $u^\mu=0$ we obtain:

$$\widetilde{\Gamma}^\gamma_{\kappa\lambda}(u)=\Gamma^\gamma_{\kappa\lambda}+A^\gamma_{\kappa\lambda}$$

Choosing $A^\gamma_{\kappa\lambda}=-\Gamma^\gamma_{\kappa\lambda}$ we set the affine connection to zero at point P . $g_{\mu\nu}$ is a symmetric matrix, that can always be diagonalized, and one always obtains (in the physical case) three eigenvalues of the same sign and one opposite to the others.[2] Rescaling them one by one with $x^\mu\to\frac{\varepsilon(\mu)}{\sqrt{|\lambda|}}x^\mu$ with $\varepsilon(\mu)=-1$ and $\varepsilon(i)=1$ we get the Minkowski form of $g_{\mu\nu}$. Since the performed coordinate transformation is linear, $\Gamma^\gamma_{\kappa\lambda}$ remains zero according to its above tranformation properties.
It is clear that $\Gamma_{\alpha\beta\gamma}=0$ if and only if $\Gamma^\alpha_{\beta\gamma}=0$.

$$2\Gamma_{\alpha\beta\gamma}=\partial_\beta g_{\alpha\gamma}+\partial_\gamma g_{\alpha\beta}-\partial_\alpha g_{\beta\gamma}$$

Writing down the fully symmetric part we find

$$2\Gamma_{[\alpha\beta\gamma]}=2(\Gamma_{\alpha\beta\gamma}+\Gamma_{\beta\gamma\alpha}+\Gamma_{\gamma\alpha\beta})=\partial_\beta g_{\alpha\gamma}+\partial_\gamma g_{\alpha\beta}+\partial_\alpha g_{\beta\gamma}$$

which enables us to express the derivative of the metric: $\partial_\alpha g_{\beta\gamma}=\Gamma_{[\alpha\beta\gamma]}-\Gamma_{\alpha\beta\gamma}$ that is the derivatives of the metric vanish as long as the affine connection does.
The transformations are the linear transformations which leave the Minkowski

[2]The signature doesn't flip on a connected manifold which is proven in differential geometry by showing the existance of a smooth orthonormal set of basisvectors on a sufficiently small neighbourhood of any point. Nevertheless, in general it does not come from any set of coordinates!

metric invariant, that is the Lorentz group $SO(1,3)$.
Let's write down the Taylor expansion of the metric for a neighbourhood of a point P

$$g_{\mu\nu}(x+\Delta x) = g_{\mu\nu}(x) + \partial_\alpha g_{\mu\nu}(x)\Delta x^\alpha + \frac{1}{2}\partial_\alpha\partial_\beta g_{\mu\nu}(x)\Delta x^\alpha \Delta x^\beta = \eta_{\mu\nu} + \frac{1}{2}\partial_\alpha\partial_\beta g_{\mu\nu}(0)x^\alpha x^\beta$$

where the last line we plugged in all the input of a geodesic coordinate system verified above.

18. The metric is

$$g_{\mu\nu} = \begin{pmatrix} 1-\left(\frac{3}{13}\right)^2 & -\left(\frac{3}{13}\right)\left(\frac{4}{13}\right) & -\left(\frac{12}{13}\right)\left(\frac{3}{13}\right) \\ -\left(\frac{3}{13}\right)\left(\frac{4}{13}\right) & 1-\left(\frac{4}{13}\right)^2 & -\left(\frac{4}{13}\right)\left(\frac{12}{13}\right) \\ -\left(\frac{12}{13}\right)\left(\frac{3}{13}\right) & -\left(\frac{4}{13}\right)\left(\frac{12}{13}\right) & 1-\left(\frac{12}{13}\right)^2 \end{pmatrix}$$

Its determinant is zero, therefore (at least) one coordinate is not independent.Throwing away the z coordinate we get :

$$ds^2 = dx^2 + dy^2 - \left(\frac{3}{13}dx + \frac{4}{13}dy\right)^2$$

The determinant can be calculated again, it is nonzero. It is not difficult to find a coordinate transformation which gives the Euclidean metric $ds^2 = d\zeta^2 + d\eta^2$:

$$\zeta = \frac{12}{5}\left(\frac{3}{13}x + \frac{4}{13}y\right)$$
$$\eta = \frac{13}{5}\left(-\frac{4}{13}x + \frac{3}{13}y\right)$$

19. One can check the statemants explicitly. The scalar product of an arbitrary vector v^μ - projected according to the exercise - with u^μ gives:

$$(g_{\mu\nu} + \varepsilon u_\mu u_\nu)v^\nu u^\mu = uv(1-\varepsilon u^2)$$

which vanishes for the timelike and for the spacelike case.
If $P_{\mu\nu}$ is a projection operator orthogonal to a null vector k_μ (For example one may check that $g_{\mu\nu} - \frac{w_\mu k_\nu}{w_\alpha k^\alpha}$ is one with arbitrary vector w_μ which is not orthogonal to k_μ), then so is $P_{\mu\nu} + \alpha k_\mu k_\nu$ since $k_\mu k_\nu v^\nu k^\mu = (kv)(k^2) = 0$ since $k^\mu$is null.

20. The embedded metric in the coordinates r and ϕ can be calculated directly from the relations for the stereographic projection pluging in $\xi^1 = r\cos\phi$ and $\xi^2 = r\sin\phi$:

$$x^0 = \frac{r^2-1}{r^2+1}, \quad x^1 = \frac{2r\cos\phi}{r^2+1}, \quad x^2 = \frac{2r\sin\phi}{r^2+1}$$

The partial derivatives read:

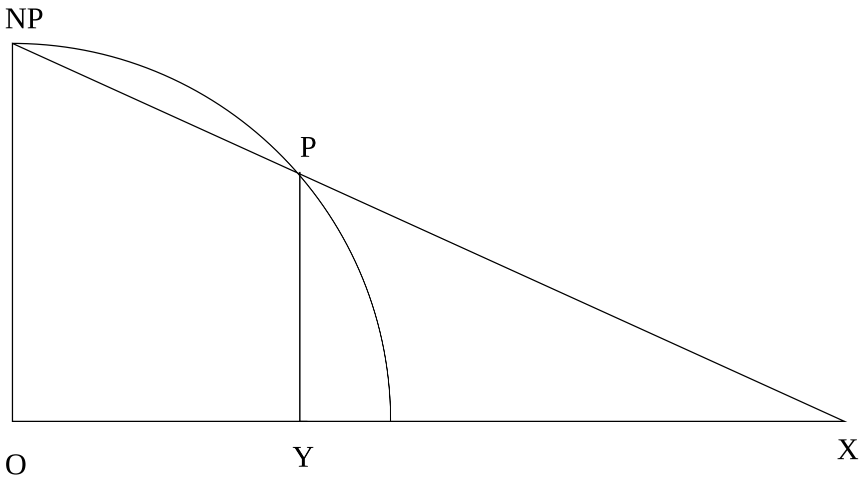
$$\begin{pmatrix} \partial_r x^0 & \partial_r x^1 & \partial_r x^2 \\ \partial_\phi x^0 & \partial_\phi x^1 & \partial_\phi x^2 \end{pmatrix} = \begin{pmatrix} \frac{4r}{(r^2+1)^2} & \frac{2(1-r^2)\cos\phi}{(r^2+1)^2} & \frac{2(1-r^2)\sin\phi}{(r^2+1)^2} \\ 0 & \frac{-2r\sin\phi}{r^2+1} & \frac{2r\cos\phi}{r^2+1} \end{pmatrix}$$

and this in deed leads to

$$g_{rr} = (\partial_r x^a)^2 = \frac{4}{(r^2+1)^2}, \quad g_{\phi\phi} = \frac{4r^2}{(r^2+1)^2}, \quad g_{\phi r} = g_{r\phi} = 0$$

which is another way of saying that the line element is

$$ds^2 = \frac{4}{(r^2+1)^2}(dr^2 + r^2 d\phi^2).$$



Let us see if the given formulae for $x^a(\xi^i)$ agree with the geometric picture, it is really the stereographic projection. We take a general point P on the sphere (x^0, x^1, x^2) and see how to determine them from the point (ξ^1, ξ^2) which is the intersection of the straight line and the plane $x^0 = 0$ where the line is determined by the North Pole (Np from now on) and P. The figure shows the plane determined by the straight line linking Np with P and its projection to the plane $x^0 = 0$. We denote the distances of the figure OY by l and OX by L. The coordinates of X o the plane are (ξ^1, ξ^2). It is clear that $l = \sqrt{(x^1)^2 + (x^2)^2}$ and $L = \sqrt{(\xi^1)^2 + (\xi^2)^2}$. From similarity of the triangles $NpOX$ and PYX :

$$x^0 = \frac{L-l}{L}$$

and we also know that $\alpha \equiv \frac{x^1}{\xi^1} = \frac{x^2}{\xi^2} = \frac{l}{L}$. α is easily determined by the equation $(x^0)^2 + (x^1)^2 + (x^2)^2 = 1$:

$$\alpha = \frac{2}{L^2+1}$$

The form of the inverse is simple now since $x^0 = 1 - \alpha$ we immediately get:

$$\xi^i = \frac{x^i}{1-x^0}, \quad i = 1, 2. \tag{10}$$

The other coordinate patch can be the stereographic projection from the South Pole, which is given by multiplying the formula for x^0 with -1

corresponding to the stereographic projection to the North Pole and those for $x^a, a = 1,2$ remain the same. The coordinate transformation relating the two patches are also easy since (10) for the South Pole projection differs also only in the sign of x^0 and one plugs the form of $x^0(\xi)$ straightly to the expression $\xi^i_{Sp}(\xi^i_{Np})$ which is made independent of x^1 and x^2. The result is the so called inversion :

$$\xi^i_{Sp} = \frac{\xi^i_{Np}}{(\xi^1_{Np})^2 + (\xi^2_{Np})^2},\ i = 1,2$$

which we may have already suspected from the geometric picture.
We prove the statement in the hint. Let us write down the Euler-Lagrange equations for the system:

$$L(x^\mu,\, \dot{x}^\nu,\, t) = \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu$$

$$\partial_\alpha L = \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{d}{dt}(g_{\alpha\nu}\dot{x}^\nu) = \partial_\mu g_{\alpha\nu}\dot{x}^\nu \dot{x}^\mu \quad + g_{\alpha\nu}\dot{\dot{x}}^\nu$$

The Euler-Lagrange equations after using the trick

$$\partial_\mu g_{\alpha\nu}\dot{x}^\nu\dot{x}^\mu = 1/2\,(\partial_\mu g_{\alpha\nu}\dot{x}^\nu\dot{x}^\mu + \partial_\nu g_{\alpha\mu}\dot{x}^\nu\dot{x}^\mu),$$

that is interchanging the summation index in half of the expression, reads

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^\alpha} - \frac{\partial L}{\partial x^\alpha} = g_{\alpha\nu}\ddot{x}^\nu + \frac{1}{2}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})\dot{x}^\mu\dot{x}^\nu$$

which is precisely the geodesic equation (in lower index) therefore we may use it for reading the Christoffel symbols off from the term quadratic in $\dot{x}^\mu$. We need to write down the Lagrangian now for our case:

$$L = \frac{4(\dot{r}^2 + r^2\dot{\phi}^2)}{(1+r^2)^2}$$

which gives

$$\Gamma^r_{rr} = -\frac{2r}{1+r^2} \quad \Gamma^r_{\phi r} = \Gamma^r_{r\phi} = 0 \quad \Gamma^r_{\phi\phi} = \frac{r(r^2-1)}{1+r^2}$$

reading off the coefficients of the terms containing $\dot{r}^2$, $\dot{r}\dot{\phi}$, $\dot{\phi}^2$ respectively of the r component of the Euler-Lagrange equation. Doing the same for the ϕ component we get:

$$\Gamma^\phi_{rr} = \Gamma^\phi_{\phi\phi} = 0, \quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1-r^2}{r(1+r^2)}.$$

The curve $r(u) = \tan\frac{u}{2}$, ϕ =constant is geodesic as it can either be explicitly checked by substituting it into the geodesic equation, but since $\Gamma^\phi_{rr} = 0$ the ϕ component of the equation is identically zero which garanties that

the geodesic equation for a generic parametrisation $\dot{x}^\mu \nabla_\mu \dot{x}^\nu = \lambda(x)\dot{x}^\nu$ (not the affine!) is surely satisfied.
For computing the paralell transport of a vector from the point with coordiantes $(R,\, \Phi_i)$ to $(R,\, \Phi_f)$ along the curve r =constant we have to solve the differential equation

$$\dot{x}^\mu D_\mu v^\nu = 0$$

We may parametrize the curve in the following way : $\phi(u) = \Phi_i + (\Phi_f - \Phi_i)u,\ u \in [0,\, 1]$. The explicit for of the differential equations are:

$$\begin{aligned} \dot{v}^r + \Gamma^r_{\phi\phi}(\Phi_f - \Phi_i)v^\phi &= 0 \\ \dot{v}^\phi + \Gamma^\phi_{\phi r}(\Phi_f - \Phi_i)v^r &= 0 \end{aligned}$$

because $\dot{r} = 0$ $\dot{\phi} = \Phi_f - \Phi_i$ and other possible terms in the summation cancel because the corresponding Γ‘ s are zero. So the nonzero Γ‘s depend only on r which is constant along the curve; we are given the differential equation for the harmonic oscillator

$$\begin{aligned} \dot{v}^r + c_1 \dot{v}^\phi &= 0 \\ \dot{v}^\phi + c_2 \dot{v}^r &= 0 \end{aligned}$$

the detailed solution of which is known or to be solved by the reader.
The only nontrivial geodesic equation is

$$\frac{R(R^2-1)}{1+R^2}\phi^2 = 0$$

which is satisfied with $R = 1$.
The surface integral is the following calculation

$$\int_0^{2\pi}\int_0^R \sqrt{g}drd\phi = \int_0^R \frac{8\pi r}{(1+r^2)^2} = 4\pi \int_0^R \frac{d(1+r^2)}{(1+r^2)^2} = \\ -\frac{4\pi}{1+r^2}\bigg|_0^R = \frac{4\pi R^2}{1+R^2}$$

The complete sphere is caracterized by $\phi \in [0,\, 2\pi]\quad r \in [0, \infty]$ (apart from one single point which does not matter concerning the integral) Taking the limit $R \to \infty$ we get for the surface 4π as was expected from the sphere with the embedding metric.

21.

$$\widetilde{\nabla}_\mu v_\nu = \partial_\mu v_\nu - \widetilde{\Gamma}^\alpha_{\mu\nu} v_\nu$$

$$\widetilde{\Gamma}^\alpha_{\mu\nu} = \frac{1}{2\Omega^2} g^{\alpha\beta}\left(\partial_\mu(\Omega^2 g_{\beta\nu}) + \partial_\nu(\Omega^2 g_{\beta\mu}) + \partial_\beta(\Omega^2 g_{\mu\nu})\right) = \\ \Gamma^\alpha_{\mu\nu} + \frac{1}{\Omega}(\delta^\alpha_\nu\, \partial_\mu\Omega + \delta^\alpha_\mu \partial_\nu\Omega - g^{\alpha\beta} g_{\mu\nu}\partial_\beta\Omega)$$

where the factor Ω drops out of the terms $\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}$ (that is when the partial derivative acts on g) which gives precisely $\Gamma_{\beta\mu\nu}$. So the tensor $C^\alpha_{\mu\nu}$ can be read off now:

$$C^\alpha_{\mu\nu} v_\alpha = \nabla_\mu v_\nu - \widetilde{\nabla}_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\alpha_{\mu\nu} v_\alpha - \partial_\mu v_\nu + \widetilde{\Gamma}^\alpha_{\mu\nu} = \\ \frac{1}{\Omega}(\delta^\alpha_\nu\, \partial_\mu\Omega + \delta^\alpha_\mu \partial_\nu\Omega - g^{\alpha\beta} g_{\mu\nu}\partial_\beta\Omega) v_\alpha$$

- Sufficiency is easy, since multiplying the metric with a number means that the norm $(g_{\mu\nu}v^\mu v^\nu)$ of a vector v^μ will be a positive number times the original norm, thus the causal structure (that is the norm being positive, negative or zero corresponding to spacelike, timelike and null vector, respectively) is preserved.
  It is also necessary, because: Consider a point P and the metric in the geodesic coordinate system with $g_{\mu\nu}(P) = \eta_{\mu\nu}$. Take the vectors $E^\mu_{(\alpha)} = \delta^\mu_\alpha$ that is precisely that orthonormal set in which the metric is Minkowski. The null vectors $E^\mu_{(0)} \pm E^\mu_{(i)}$ is supposed to be null also with respect to $\widetilde{g}_{\mu\nu}$. The norm of the timelike vector should be negative by assumption with respect to $\widetilde{g}_{\mu\nu}$.

$$0 > \widetilde{g}_{\mu\nu}E^\mu_{(0)}E^\nu_{(0)} = \widetilde{g}_{\mu\nu}\delta^\mu_0\delta^\nu_0 = \widetilde{g}_{00} \tag{11}$$

$$0 = \widetilde{g}_{\mu\nu}(E^\mu_{(0)} \pm E^\mu_{(i)})(E^\nu_{(0)} \pm E^\nu_{(i)}) = \widetilde{g}_{00} + \widetilde{g}_{ii} \pm 2\widetilde{g}_{i0}$$

  which means

$$\widetilde{g}_{i0} = \widetilde{g}_{0i} = 0$$

  and

$$\widetilde{g}_{ii} = -\widetilde{g}_{00}. \tag{12}$$

  Finally the nullity of the vector $E^\mu_{(0)} + (E^\mu_{(i)} + E^\mu_{(j)})/\sqrt{2} \quad (i \neq j)$ gives for $\widetilde{g}_{\mu\nu}$:

$$0 = \widetilde{g}_{00} + \frac{\widetilde{g}_{ii}}{2} + \frac{\widetilde{g}_{jj}}{2} + \widetilde{g}_{ij} = \widetilde{g}_{ij}$$

  since the spatial diagonal components are all equal to $-\widetilde{g}_{00}$ by (12). Hence

$$\widetilde{g}_{\mu\nu} = diag(-a, a, a, ..., a) = a g_{\mu\nu}$$

  with $a = -\widetilde{g}_{00}$ which is positive by (11)

- We can write explicitly the equation of the geodesic not in the affine parametrization.

$$\frac{dx^\nu}{d\lambda}\frac{\partial}{\partial x^\nu}\frac{dx^\mu}{d\lambda} + \frac{dx^\nu}{d\lambda}\Gamma^\mu_{\nu\alpha}\frac{dx^\alpha}{d\lambda} = \alpha\frac{dx^\mu}{d\lambda} \tag{13}$$

  which coincides with (5.8a) of the lecture notes and according to exercise 11. we can take a parametrisation where $d^2\lambda'/d\lambda'^2 = \alpha$ and the right hand side of (13) drops out.

- Null geodesics have null tangent whose norm is zero even after multiplying it (the metric) with a number.

-

$$\begin{aligned}\widetilde{g}^{\mu\nu}\widetilde{D}_\mu\widetilde{D}_\nu\widetilde{\Psi} = \widetilde{g}^{\mu\nu}\widetilde{D}_\mu\partial_\nu(\Omega^s\Psi) = \widetilde{g}^{\mu\nu}\widetilde{D}_\mu(s\Omega^{s-1}\Psi\partial_\nu\Omega + \Omega^s\partial_\nu\Psi) = \\ \widetilde{g}^{\mu\nu}(s(s-1)\Omega^{s-2}\Psi\partial_\mu\Omega\partial_\nu\Omega + s\Omega^{s-1}\partial_\mu\Psi\partial_\nu\Omega + s\Omega^{s-1}\Psi D_\mu\partial_\nu\Omega + \\ + s\Omega^{s-1}\partial_\mu\Omega\partial_\nu\Psi + \Omega^s D_\mu\partial_\nu\Psi) - \widetilde{g}^{\mu\nu}C^\alpha_{\mu\nu}(s\Omega^{s-1}\Psi\partial_\alpha\Omega + \Omega^s\partial_\alpha\Psi) = \\ = \Omega^{s-2}g^{\mu\nu}D_\mu D_\nu\Psi + \Omega^{s-3}\left((2s+d-2)g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Omega + s\Psi D_\mu\partial_\nu\Psi\right) + \\ \Omega^{s-4}s(s+d-3)\Psi g^{\mu\nu}\partial_\mu\partial_\nu\Omega\end{aligned}$$

The definition of the (conformally) transformed quantities (with the tilde) are given in the exercise. Both the original and the transformed covariant derivatives acts the same way (namely as ordinary derivatives) when acting on a scalar; these give the first five terms in the first equality. When $\widetilde{D}_\mu$ acts on a vector it acts as given in first part of this exercise, the tensor $C^\alpha_{\mu\nu}$ emerges. One plugs in its explicit expression and finally collects the terms which has Ω on the same power then one obtains the last equality, where d is the dimension of the space emerging from $g_{\mu\nu}g^{\mu\nu} = \delta^\mu_\mu = d$. Therefore the Klein Gordon equation (with zero mass) is conformally invariant if all terms vanishes in the last equation but the first: s must be chosen to be zero and $d = 2$ is the only possibility.

- For the Maxwell equations we are led to the following expression (with some intermediate steps omitted):

  $$\widetilde{D}^\mu \widetilde{F}_{\mu\alpha} = \widetilde{g}^{\mu\nu}\widetilde{D}_\nu \widetilde{F}_{\mu\alpha} = \Omega^{s-2}g^{\mu\nu}D_\nu F_{\mu\alpha} + (d-4+s)\Omega^{s-3}g^{\mu\nu}F_{\mu\alpha}\partial_\nu\Omega$$

  and
  $$\widetilde{D}_{[\alpha}\widetilde{F}_{\beta\gamma]} = \Omega^s D_{[\alpha}F_{\beta\gamma]} + s\Omega^{s-1}D_{[\alpha}\Omega F_{\beta\gamma]}$$
  where the indices in the brackets [ ] are totally symmetrized.[3]The requirement for conformal invariance is

  $$\left.\begin{array}{c} d-4+s=0 \\ s=0 \end{array}\right\} \Longrightarrow d=4$$

  that is the Maxwell equations are conformally invariant in four dimensions with zero conformal dimension if the field strength.

- The transformed equation for the symmetric stress-energy tensor after expressing all terms explicitly and collecting the same powers of Ω as above :

  $$\widetilde{D}_\mu \widetilde{T}^{\mu\nu} = \Omega^s D_\mu T^{\mu\nu} + \Omega^{s-1}\left((s+d-2)T^{\mu\nu}D_\mu\Omega - g^{\nu\mu}TD_\mu\Omega\right)$$

  where the last terms vanises only if

  $$s = -d+2$$

  and $T = 0$. Thus conformal invariance holds. Thus the stress-energy tensor of a conformal invariant field must be traceless (since the conservation equation should hold for the transformed quantities too).

22. Since $g_{\mu\nu}$ is dimensionless $[\Lambda] = [\Lambda g_{\mu\nu}] = [R_{\mu\nu}] = [\partial^2 g_{\mu\nu}] = m^{-2}$.
One needs to use the Newtonian limit of general relativity, which is defined by

    (a) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is "small".

    (b) $h_{\mu\nu}$ is slowly varying: $\partial_0 h_{\mu\nu} \approx 0$.

    (c) The derivatives $\partial_i h_{\mu\nu}$ are of the order $h_{\mu\nu}$.

[3]For the definition see exercise 8.

(d) Particles move slowly $u^\mu = (c, 0, 0, 0)$.

(e) $T_{00} \equiv -\rho$ is the only nonzero component of the stress-energy tensor (therefore $T = -\rho$ holds as well, current densities and tidal forces vanish in the Newtonian limit.

The geodesic equation then reduces to $\frac{d^2x^i}{dt^2} = -\frac{1}{2}\partial_i h_{00}$ thus the gravitational potential is $\Phi = h_{00}/2$. Due to (c) only the "$\partial\Gamma$ terms remains in the Riemann tensor in the Newtonian limit; for the zeroth component of the Ricci tensor only the first term survives $R_{00} = \partial_i \Gamma^i_{00} = \Delta h_{00}/2$. Taking the trace of the Einstein equation we get $R = 4\Lambda - \kappa T$ where we use the shorthand notation $\kappa = -\frac{8\pi G}{c^4}$. Plugging it back to Einstein's equation for the $_{00}$ component yields:

$$\Delta\Phi - \frac{1}{2}(2\Lambda - \kappa T)(-1 + 2\Phi) = \kappa T$$

where still $O(h^2)$ terms may be present. Setting first $\Lambda = 0$ we get $\Delta\Phi = \frac{1}{2}\kappa T(3 - \Phi)$ where it is clear that the second term can be dropped since κT should be already of order Φ ($\sim h_{00}$) in our approximation. Thus putting back Λ considering the term

$$\rho' = -\frac{\Lambda g_{00}}{\kappa} \tag{14}$$

as contribution to the energy density and noticing again that in $-\frac{\Lambda(-1+2\Phi)}{\kappa}$ the second term will be negligible:

$$\Delta\Phi = \kappa\left(-\frac{3\rho}{2} + \frac{\Lambda}{\kappa}\right) \tag{15}$$

One then needs to solve the Laplacian equation for the second term taking spherical symmetry into account:

$$\Delta_{radial}\phi = \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) = \Lambda$$

which admits the general solution

$$r\phi = \frac{\Lambda}{6}r^3 + \beta r + \gamma$$

β is an irrelevant additive constant of the potential for $\gamma = -GM_\odot$ from the requirement to reproduce the Newtonian potential (which is the first (source) term in (15). So the potential is

$$\phi = \frac{\Lambda}{6}r^2 - \frac{GM_\odot}{r}$$

Let's restrict the value of Λ from the requirement that in the solar system the potential is described well by the Newtonian part; that cosmological part should not exceed its value within the radius of the Pluto:

$$|\frac{\Lambda}{6}r^2_{Pluto}| < \frac{GM_\odot}{r_{Pluto}}$$

$$|\Lambda| < \frac{6GM_\odot}{r^3_{Pluto}} = 4.2 \times 10^{-35} m^{-2}$$

Finally the vacuum energy density (14) corresponding to Λ is

$$\rho' = \frac{\Lambda c^2}{8\pi G} c^2 = 2.25 \times 10^{-6} \frac{kg}{m^3} \times c^2$$

where the number is the mass density; it is much bigger than the observed mass density of the universe. Λ, if not zero, seems to be extremely small.

23. The formula

$$\widetilde{R}^{\nu}_{\kappa\lambda\alpha} = R^{\nu}_{\kappa\lambda\alpha} + D_\lambda \delta\Gamma^{\nu}_{\kappa\alpha} - D_\alpha \delta\Gamma^{\nu}_{\kappa\lambda} \qquad (16)$$

comes from formula (5.27) by plugging in $\Gamma^{\alpha}_{\beta\gamma} \to \Gamma^{\alpha}_{\beta\gamma} + \delta\Gamma^{\alpha}_{\beta\gamma}$ and keeping only the $O(\delta\Gamma)$ terms. Then one plugs in the expression for the variation of the connection:

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(D_\mu g_{\alpha\nu} + D_\nu g_{\alpha\mu} - D_\alpha g_{\mu\nu})$$

into (16), uses the fact the the $D_\mu g_{\nu\rho} = 0$ and arrives immediately at formula (8.7) of the lecture notes:

$$\widetilde{R}_{\mu\nu} = \widetilde{R}^{\alpha}_{\mu\alpha\nu} = R_{\mu\nu} + \frac{1}{2}(-g^{\alpha\beta} D_\alpha D_\beta \delta g_{\mu\nu} + D_\alpha D_\mu \delta g^{\alpha}_{\nu} + D_\alpha D_\nu \delta g^{\alpha}_{\mu} - D_\mu D_\nu \delta g^{\alpha}_{\alpha})$$

where $\delta g^{\alpha}_{\beta} = g^{\alpha\mu}\delta g_{\mu\beta}$.

24. We shall use the identity given in the hint. Let us also use the shorthand $g = -\det(g_{\mu\nu})$

$$\delta(\sqrt{g}) = \frac{1}{2\sqrt{g}}\delta g = -\frac{1}{2\sqrt{g}}\delta \exp Tr\,(\log(g_{\mu\nu})) = -\frac{g}{2\sqrt{g}} Tr \delta \log(g_{\mu\nu}) =$$
$$= -\frac{\sqrt{g}}{2} Tr(g^{\mu\alpha}\delta g_{\alpha\nu}) = -\frac{\sqrt{g}}{2} g^{\mu\nu}\delta g_{\mu\nu} = \frac{\sqrt{g}}{2} g_{\mu\nu}\delta g^{\mu\nu}$$

We used the mentioned identity in the second eqation. Note that Tr is just the ordinary trace of a matrix and also $\delta \log A = A^{-1}A$. In the last equality we use

$$0 = \delta(g^{\mu\nu}g_{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu} + g_{\mu\nu}\delta g^{\mu\nu}.$$

The identity

$$\exp Tr A = \det \exp A$$

is still to be proven. Suppose A is in an upper triangular form. Then

$$\det \exp A = \prod_i \exp A_{ii} = \exp \sum_i A_{ii} = \exp Tr A$$

since any power of a matrix A of upper triangual form is of the same form so is $\exp A$ then.

25. The variation of the action is

$$\delta S = \int d^4x \left(\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \partial_\mu \phi}\delta\partial_\mu\phi + \frac{\partial L}{\partial x^\mu}\delta x^\mu \right) + \int \delta(d^4x) L$$

The last term is the variation of the integration measure due to the coordinate change. In the second term we perform the usual integration by part:

$$\int_V d^4x \frac{\partial L}{\partial\partial_\mu\phi}\delta\partial_\mu\phi = \int_V d^4x \left(\partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi}\delta\phi\right) - \partial_\mu\frac{\partial L}{\partial\partial_\mu\phi}\delta\phi\right)$$

We need to determine the Jacobian for the transformation $x^\mu \to x^\mu + \delta x^\mu$ to the first order in δx^μ. $\partial x'^\mu/\partial x^\nu = \delta^\mu_\nu + \partial(\delta x^\mu)/\partial x^\nu$ When calculating the determinant of this expression the nondiagonal elements are all of first order, thus we have to consider (from the sum

$$\det M = (1/24)\,\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\pi\rho\gamma\delta}M_{\mu\pi}M_{\nu\rho}M_{\alpha\gamma}M_{\beta\delta}$$

for the determinant of the matrix M) only at least three "1"s (of four) diagonal terms in a generic member of the above sum, but then we have no other choice for the forth term than the remaining diagonal one. Therefore

$$\delta(d^4x) = \left(\left|\frac{\partial x'^\mu}{\partial x^\nu}\right| - 1\right) d^4x = \partial_\mu\delta x^\mu d^4x$$

So rearranging slightly our terms for the variation and taking into account, that $(\partial_\mu L)\delta x^\mu + L\partial_\mu x^\mu = \partial_\mu(L\delta x^\mu)$:

$$\delta S = \int d^4x \left(\partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\phi}\delta\phi + L\delta x^\mu\right) + \left(\frac{\partial L}{\partial\phi} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\phi}\right)\delta\phi\right)$$

which is the desired form of the variation with the divergence of the Noether current

$$F^\mu = \frac{\partial L}{\partial\partial_\mu\phi}\delta\phi + L\delta x^\mu \tag{17}$$

and the equations of motion multiplied with $\delta\phi$.
The Noether current for translation of the free scalar field is easily determined by substituting the Lagrangian into (17)

$$L = \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right)$$

The variation of the coordinate $\delta x^\mu = a^\mu$ where a^μ is infinitesimal constant displacement. Note also that

$$\phi'(x^{\mu\prime}) = \phi(x'^\mu - \delta x^\mu) = \phi(x'^\mu) - \partial_\mu\phi(x'^\nu)a^\mu \equiv \phi(x'^\mu) + \delta\phi(x'^\mu)$$

that is the corresponding variation of the field is $\delta\phi = -\partial_\mu\phi a^\mu$. We can now write down the expression for the current

$$F^\mu_\nu a^\nu = \partial^\mu\phi\partial_\nu\phi a^\nu - \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right)a^\nu$$

where we separated the infinitesimal parameter. The current is thus the stress-energy tensor as expected[4]

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left((\partial\phi)^2 + m^2\phi^2\right)g_{\mu\nu}$$

[4]The sign difference from the formula in the lecture notes arises only due to the different conventions for defining the stress energy by Hilbert's prescription via the variation of the action with respect to the variation of the metric and by identifying it with the Nother current associated to translational symmetry.

Regarding the currents see exercise 1.

26. The covariant derivative for a two low index tensor reads $D_\alpha F_{\beta\gamma} = \partial_\alpha F_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} F_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} F_{\beta\delta}$. Now the deviation from the flat space Bianchi identity is:$-\left(\Gamma^\delta_{\alpha\beta} F_{\delta\gamma} + \Gamma^\delta_{\alpha\gamma} F_{\beta\delta} + \Gamma^\delta_{\beta\gamma} F_{\delta\alpha} + \Gamma^\delta_{\beta\alpha} F_{\gamma\delta} + \Gamma^\delta_{\gamma\alpha} F_{\delta\beta} + \Gamma^\delta_{\gamma\beta} F_{\alpha\delta}\right)$ where the terms can be matched as (1.-4., 2.-5., 3.-6.); they vanish since $F_{\alpha\beta}$ is antisymmetric. Since the ordinary and the covariant form of the Bianchi identity are thus equivalent in one coordinate system, the covariant form is valid and meaningful in curved space.
For the second part we wish to prove for a quantity A which is either an antisymmetric upper index tensor or a contravector

$$D_\mu A^{\mu\{\nu\}} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} A^{\mu\{\nu\}}\right) \tag{18}$$

We start working out both sides:

$$\partial_\mu A^{\mu\{\nu\}} + \Gamma^\mu_{\mu\alpha} A^{\alpha\{\nu\}} + \left\{\Gamma^\nu_{\mu\alpha} A^{\mu\alpha}\right\} = \partial_\mu A^{\mu\{\nu\}} - \frac{1}{2g}(\partial_\mu g) A^{\mu\{\nu\}}$$

Subtracting the ordinary derivative from both sides and noticing that the third term on the right hand side is identically zero since Γ is symmetric and A is antisymmetric, we get:

$$\Gamma^\mu_{\mu\alpha} A^{\alpha\{\nu\}} = \frac{1}{2g}(\partial_\mu g) A^{\mu\{\nu\}}$$

Now we rename the indices so as to omit A from the equation and rewrite the derivative on the right hand side. The result is:

$$\Gamma^\beta_{\beta\mu} = \frac{1}{2} \partial_\mu \log(-g) \tag{19}$$

Writing down the left hand side explicitly in terms of the metric one observes that the last two terms cancel and one is lead to:

$$\Gamma^\beta_{\beta\mu} = \frac{1}{2} g^{\beta\alpha} \partial_\mu g_{\alpha\beta} \tag{20}$$

Now as a last step we use the statement of the hint in exercise 24 for a generic matrix[5] :

$$\partial_\mu \log \det A = \partial_\mu tr \log A = tr(A^{-1} \partial_\mu A) = A^{-1}_{\beta\alpha} \partial_\mu A_{\alpha\beta}.$$

Now substituting g in the place of A we find the desired equation equating the right hand side of (20) and (19)[6] So (18) holds for $F^{\mu\nu}$ and differentiating it again gives identically zero due to the the symmetry of $\partial_\mu \partial_\nu$ and the antisymmetry of $F^{\mu\nu}$.

$$0 = \partial_\mu \partial_\nu (\sqrt{-g} F^{\mu\nu}) = -\partial_\nu (\sqrt{-g} J^\nu)$$

which is formula (10.14) of the lecture notes.

[5] Here we use the logarithm of that equation with $\exp A \to A$ substitution.

[6] The minus sign does not matter: symbolicly $(\log -f)' = (\log f)' = f'/f$. Of course only the positive term makes sense.

27. According to the hint all we need to do is Lorenz-transforming back from the rest frame stress-energy tensor to the inertial frame. The nonzero components after the Lorentz transformation are (our velocity parameter is $-v$ then):

$$T^{00} = -\rho_0\gamma^2 \quad T^{x0} = T^{0x} = -\rho_0\gamma^2 v \quad T^{xx} = -\rho_0 v^2\gamma^2$$

where $\gamma = \sqrt{1-v^2}$.
According to the hint we have exactly the same formulae as above using ϕ instead of x and $v = ar$. We need to relate ρ_0 now to the collection of particles. Let us write down the total energy of the particles:

$$T^{00} 2\pi dr dz = \gamma N m$$

with N being the number m the mass of particles. In the continuum approximation ρ_0 is therefore:

$$\rho_0 = \frac{Nm\delta(r-a)\delta(z)}{2\pi a\gamma}$$

Reexpressing the result to the rectangular coordinates according to $x = r\cos\phi \quad y = r\sin\phi$ we find

$$T^{\mu\nu} = -\frac{\gamma Nm\delta(\sqrt{x^2+y^2}-a)\delta(z)}{2\pi a}\cdot$$
$$\cdot\begin{pmatrix} 1 & -v\sin\phi & v\cos\phi & 0 \\ -v\sin\phi & v^2\sin^2\phi & -v^2\sin\phi\cos\phi & 0 \\ v\cos\phi & -v^2\sin\phi\cos\phi & v^2\cos^2\phi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

If we have an additional ring rotating in the opposite direction then the stress-energy is the same as above with the sign of v reversed. In the sum terms linear in v disappear; terms quadratic in v are doubled. In polar coordinates the nonzero components are

$$T^{00} = \frac{1}{v^2}T^{\phi\phi} = -\frac{\gamma Nm\delta(r-a)\delta(z)}{\pi a}.$$

The continuity equation refers to the conservation of energy, but since we do not deal with a closed system, it simply does not hold. We need to encounter the forces which make the non-inertial motion possible for a closed system to be given in which energy is conserved; the associated $T^{\mu\nu}$ is divergencefree ($\partial_\mu T^{\mu\nu}$).

28. In the previous problem we saw that for particles with four-velocity u^μ, $T^{\mu\nu} \sim u^\mu u^\nu$. For an isotropic collection with $u^\mu = \gamma(1, v\vec{n})$ we have $T^{\mu\nu} = -\kappa < u^\mu u^\nu >$ where κ is a positive constant and the average is taken over the orientations of $\vec{n}$. By symmetry, off-diagonal components average to zero

$$T^{0i} = T^{i0} = -\kappa\gamma^2 v < n^i >= 0 \quad T^{ij} = -\kappa\gamma^2 v^2 < n^i n^j >= 0 \; i \neq 0.$$

To evaluate the diagonal spatial components note that $< n^x n^x >=< n^y n^y >=< n^z n^z >$ and that $< n^x n^x > + < n^y n^y > + < n^z n^z >= 1$ so that $< n^i n^i >= 1/3$,

$$T^{ij} = \frac{-\kappa\gamma^2 v^2}{3}.$$

Each moving particle has energy γm so the density of mass energy must be

$$m\gamma N = T^{00} = \kappa\gamma^2 < 1 >= \kappa\gamma^2$$

which gives us the value of κ. The nonzero components again are

$$T^{00} = -m\gamma N \quad T^{ij} = -\frac{m\gamma N v^2}{3}\delta_{ij}.$$

For photons $v \to c$, $m\gamma \to h\nu$. For cold dust $v = 0$ so $T^{00} = mN$ is the only nonvanishing component.

29. In the rest frame of the fluid $u = (1, \vec{0})$ so that

$$T^{00} = -pg^{\mu\nu} - (\rho + p)u^\mu u^\nu$$

Since this is a tensor equation which is true in one frame, it must be true in general.

30. If O_1 is freefalling it follows a geodesic. The only nontrivial geodesic equation is the one for r

$$\Gamma^r_{tt}\dot{t}^2 + \Gamma^r_{\phi\phi}\dot{\phi}^2 = 0$$

which gives after plugging in the connections and using $\dot{\phi}/\dot{t} = d\phi/dt = \omega$ gives a relation between ω and R:

$$\omega^2 = \frac{m}{R^3} \tag{21}$$

The possible values of ω - with any power of the rocket engine - are restricted by the fact, that the observer follows a timeslike curve $ds^2 < 0$:

$$-\left(1 - \frac{2m}{r}\right)dt^2 + r^2 d\phi^2 < 0$$

which gives

$$\omega < \frac{1}{R}\sqrt{\left(1 - \frac{2m}{R}\right)} \tag{22}$$

and, after plugging in (21) for obtaining the possible radii for freefalling motion:

$$R > 3m.$$

To find the maximal $|\omega|$ one has to solve $d\omega^2_{max}(R)/dR = 0$ where ω_{max} is the value of ω which makes (22) equality. The only solution is

$$R = 4m$$

That is since $\omega(R) > 0$, $\omega^2(2m) = \omega(\infty) = 0$ really the maximum is

$$\omega_{Max} = \frac{\sqrt{2}}{8m}$$

which may be approached (but not reached as corresponding to the speed of light) by a strong rocket engine. M is capitalized in the subscript in the previous formula indicating that it is the absolute maximum (not depending on R).
The last question may be reformulated by asking what the ratio of $\Delta\tau_3/\Delta\tau_2$ of differences in proper time between emitting two photons for O_2 and recieving them for O_3 since the period T of radiation of a given frequency is the wavelength. The situation is static the coordinate interval $\Delta t_1 = \Delta t_2$ for both observers and

$$d\tau^2 = -\left(1 - \frac{2m}{r}\right) dt^2$$

since they are not moving. Therefore

$$\frac{\lambda'}{\lambda} = \frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{1 - 2m/L}{1 - 2m/R}}.$$

31. We can start off with formula (12.18) $u'' + u = M/J^2 + 3Mu^2$ of the lecture notes, we are not dealing with circular orbit now. The expansion goes exactly like for the null case:

$$u = A\cos\phi + v.$$

The equation for v reads

$$v'' + v = \frac{M}{J^2} + \frac{3}{2}MA^2(1 + \cos 2\phi).$$

The solution is

$$u = A\cos\phi + \frac{M}{J^2} + MA^2(2 - \cos^2\phi).$$

The rest of the calculation amounts to solving $1/r \equiv u = 0$ for $\cos\phi$, expanding the the solution around $\pi/2$ as M is considered a small parameter in our case. Substituting $r^0 = 1/A$ at the end and writing down that the deflection angle is twice the deviation of ϕ from $\pi/2$ as is the deviation between the ingoing and the outgoing particle; very much like in the lecture notes but with the constant M/J^2 term present. The result reads:

$$\Delta_{grav} = \frac{4M}{r_0}\left(1 + \frac{r_0^2}{2J^2}\right)$$

We may slightly rewrite this expression by $J^2 = L^2/m^2 = (-E^2 + m^2)r_0^2/m^2 = r_0^2v^2/(1 - v^2)$, (where L^2 is the angular momentum[7] and $v^2$is the velocity of the particle at infinity):

$$\Delta_{grav} = \frac{2M}{r_0}\left(\frac{1 + v^2}{v^2}\right) \tag{23}$$

[7]The nonzero z component M_{xy} of the angular momentum $M_{\mu\nu} = x_\mu p_\nu - x_\mu p_\nu$ can be shown to be equal mr_0v/γ with v being the velocity of the particle at infinity.

The equation for the charge around a nucleus is with the abbreviation $\gamma = \sqrt{1-v^2}$:

$$\frac{d}{dt}\gamma m v = \gamma m r \omega^2 - \frac{Ze^2}{r^2} \tag{24}$$

and $L = m\gamma r^2\omega^2$ and $E = m\gamma - Ze^2/r$ constants of the motion. Rewriting (24) with the help of them and using the trick $d/dt = (L/m\gamma r^2)d/d\phi$ leads to for $u = 1/r$

$$u'' + \left(1 - \frac{Z^2e^4}{L^2}\right)u = \frac{Ze^2E}{L^2}$$

where u is the the funtion of ϕ. This is solved by

$$u = \frac{\cos\left(\sqrt{1 - Z^2e^4/L^2}\right)\phi}{r_0} + \frac{Ze^2E}{L^2 - Z^2e^4}$$

which for large L reduces to

$$u = \frac{1}{r_0}\cos\phi + \frac{Ze^2E}{L^2}$$

The deflection angle can now be determined as above:

$$\Delta_{em} = \frac{2r^0Ze^2E}{L^2} = \frac{2Ze^2}{mr_0}\frac{\sqrt{1-v^2}}{v^2}. \tag{25}$$

(25) and (23) differ due to the difference in the tensor nature of gravity and the vector nature of electromagnetism. The fields Lorentz transform differently and this induces a different dependence of the deflection on v when v is not small.
The bending of the trajectory of the nonrelativistic particle in the gravitational potential of a star is obtained by substituting mM in the place of Ze^2 in (25) and omitting the square root as in real units it reads $\sqrt{1-v^2/c^2} = 1$ in our case. The result is

$$\Delta_{nonrel} = \frac{2M}{r_0v^2}$$

which is $1/2$ times the result for the photon in appropriate units. (Note we used $G = 1$ in the exercise.) This can also be determined from the general integral of the trajectory of a particle moving in a potential $-GmM/r$.

32. From the general equation $r = p/(1 + e\cos\phi)$ for te ellipse $u + u'' = 1/p = M/J^2$ (from (12.18) considering the leading order of the right hand side which provides the ellipse). $p = a(1-\varepsilon^2)$ also holds between the parameter p , great axis a and excentricity ε of the ellipse. We know that $M = Gm_\odot$ and therefore

$$[M^2/J^2] = [G]^2kg^2/[J]^2 = \left(\frac{m^3}{kg\,s^2}\right)^2 kg^2\left(\frac{s}{m^2}\right)^2 = \frac{m^2}{s^2}$$

thus in order to obtain dimensionless quantity we should put c^2 in the denominator which is the unique way the recover real units. So

$$\delta\phi = 6\pi\frac{M^2}{J^2} = 6\pi\frac{M}{p\,c^2} = 6\pi\frac{m_\odot G}{a(1-\varepsilon^2)c^2}.$$

33. For the rocket ship to be moving along a timelike worldline its four-velocity must satisfy:

$$-1 = g_{\mu\nu}u^\mu u^\nu = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$$

where dot means differentiation with respect to proper time τ . Inside the horizon all of these terms are positive except the second so

$$\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 < -1$$

We also know that $\dot{r} < 0$ inside the horizon from the Kruskal coordinates $0 < dt/2M = dx/x - dy/y$, for inside of the black hole $x > 0$, $dx > 0$, thus $0 < dx/x + dy/y = dr/2M(1-2M/r)$ from which $dr < 0$ follows. Thus

$$dr < -d\tau\sqrt{\frac{2M}{r}-1}$$

and

$$\tau_{max} = -\int_{2M}^{0}\left(\frac{2M}{r}-1\right)^{-\frac{1}{2}}dr =$$

$$r^2\sqrt{2M-r} + M\arccos\left(\frac{r}{M}-1\right)\Big|_{2M}^{0} = \pi M$$

34. The solution will be identical to the electric case with Q being the charge of the magnetic monopole and $E(r) = F_{12} = -F_{21}$ the magnetic field since the Maxwell action without source $-1/4F_{\mu\nu}F^{\mu\nu}$ is invariant under the transformation [8]

$$F_{\mu\nu} \to \star F_{\mu\nu}$$

$$\star F_{\mu\nu}\star F^{\mu\nu} = \frac{1}{4}E_{\mu\nu\alpha\beta}E^{\mu\nu\gamma\delta}F^{\alpha\beta}F_{\gamma\delta} = -\frac{1}{2}(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\gamma_\beta\delta^\delta_\alpha)F^{\alpha\beta}F_{\gamma\delta} =$$
$$\frac{1}{2}(F^{\gamma\delta}F_{\gamma\delta} - F^{\delta\gamma}F_{\gamma\delta}) = F^{\gamma\delta}F_{\gamma\delta}$$

[9]by antisymmetry and therefore the corresponding stress energy tensor

$$T_{\mu\nu} = 2\frac{\delta S_{Max}}{\delta g^{\mu\nu}}$$

remains identical if expressed in the same variable. Let's see how the calculation match the electric case. The field strength should have the form $-F_{32} = F_{23} = F_{\theta\phi} = b$ (that is pointing to the radial direction); the rest of the components are zero. Now the first Maxwell eqution $\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0$ gives:

$$\partial_\theta(\sqrt{-g}F_{\theta\phi}g^{\theta\theta}g^{\phi\phi}) = \partial_\theta\frac{br^2\sin\theta\sqrt{AB})}{r^4\sin^2\theta} = 0$$

[8]See exercise 7. for the definition.

[9]Note that $E^{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}/\sqrt{-g}$ and $E_{\alpha\beta\gamma\delta} = \sqrt{-g}\varepsilon_{\alpha\beta\gamma\delta}$ in curved space.

that is the b has the form

$$b = \sin\theta \hat{b}(r).$$

The second equation $1/2\partial_\mu(\sqrt{-g}E^{\mu\nu\gamma\delta}F_{\gamma\delta}) = 0$ leads to

$$\partial_r(F_{\theta\phi}) = \partial_r b = 0$$

that is

$$b = \frac{q}{4\pi}\sin\theta$$

with q a constant. Now we write down the stress energy tensor similarly to the electric case:

$$\begin{aligned}
T_{00} &= -\frac{Ab^2}{2r^4\sin^2\theta} = -\frac{Aq^2}{32\pi^2 r^4} \\
T_{11} &= \frac{Bb^2}{2r^4\sin^2\theta} = \frac{Bq^2}{32\pi^2 r^4} \\
T_{22} &= -\frac{b^2}{2r^2\sin^2\theta} = -\frac{q^2}{32\pi^2 r^2} \\
T_{33} &= -\frac{b^2}{2r^2} = -\frac{q^2\sin^2\theta}{32\pi^2 r^2}
\end{aligned}$$

These precisely coincide with (13.9-13.12) of the lecture notes for the electric case. From this point the calculation is identical yielding (13.19) for the uknowns A and B

$$A = 1 - \frac{2M}{r} + \frac{q^2}{4\pi r^2} \quad B = \frac{1}{A}.$$

The physical magnetic field is

$$\hat{B}(r) = *F_{01} = \frac{1}{2}_{01\mu\nu}F^{\mu\nu} = \sqrt{-g}\varepsilon_{0123}F_{23}g^{22}g^{33} = \frac{q\sqrt{AB}}{4\pi r^2} = \frac{q}{4\pi r^2}$$

which is spherically symmetric and shows after integration that q is the magnetic charge.[10]

35. The Chrystoffel symbols are most easily determined via the Lagrangian principle, that yield the geodesic equations[11].

$$L = -\dot{u}\dot{v} + \frac{1}{2}L^2 e^{2\beta}\dot{x}^2 + \frac{1}{2}L^2 e^{-2\beta}\dot{y}^2$$

(a) The Euler Lagrange for this Lagrangian read:

$$\begin{aligned}
-\ddot{v} + Le^{2\beta}(L' + L\beta')\dot{x}^2 + Le^{-2\beta}(L' - L\beta')\dot{y}^2 &= 0 \\
L^2 e^{2\beta}\ddot{x} + 2Le^{2\beta}(L' + L\beta')\dot{u}\dot{x} &= 0 \\
L^2 e^{-2\beta}\ddot{y} + 2Le^{-2\beta}(L' - L\beta')\dot{u}\dot{y} &= 0 \\
\ddot{u} &= 0
\end{aligned}$$

[10] $B = g_{11}$ the rest of the "b"'s represent functions related to the magnetic field. This one above is the real physical measurable field.

[11] See exercise 20. for the details.

for the coordinates u, x, y and v, respectively. One can read off the connection:

$$\begin{aligned}
\Gamma^v_{xx} &= -Le^{2\beta}(L' + L\beta') \\
\Gamma^v_{yy} &= -Le^{-2\beta}(L' - L\beta') \\
\Gamma^x_{xu} = \Gamma^x_{ux} &= \frac{L'}{L} + \beta' \\
\Gamma^y_{yu} = \Gamma^y_{uy} &= \frac{L'}{L} - \beta'
\end{aligned}$$

and the rest are zero.

(b) The nontrivial geodesic equations are (observing that $\dot{t} = \dot{u} + \dot{v}$)

$$\begin{aligned}
\ddot{u} &= 0 \\
\ddot{v} &= 0
\end{aligned}$$

since $\Gamma^u_{\mu\nu} = 0$ for all values for μ and ν and $\Gamma^v_{\mu\nu} \neq 0$ for indices yy and xx but since $\dot{x} = \dot{y} = 0$ for the given curve, the quadratic term vanishes. Thus $\dot{u} = -\dot{v} - 1 = k$ constant satisfies the geodesic equations and it is obvioulsly timelike. $ds^2/d\tau^2 = -1$.

(c)

$$R_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}$$

All Γ-s depend on only u but $\Gamma^u_{\mu\nu}$ are zero hence the first term is zero. The third term also gives no contribution since $\Gamma^\lambda_{\lambda\mu}$ is nonzero only with $\sigma = u$, but it is contracted with $\Gamma^u_{\mu\nu}$ which is zero. Doing the sum now in the surviving two terms we get nenzero only for uu components:

$$R_{uu} = -\partial_u \Gamma^x_{ux} - \partial_u \Gamma^y_{uy} - (\Gamma^x_{ux})^2 - (\Gamma^y_{uy})^2 = 2\left(\frac{L''}{L} + \beta'^2\right)$$

(d) The Einstein equation for $\beta(u) = u$ is the one of the harmonic oscillator

$$L'' = -L,$$

with the solution (matching the continuity assumptions with $L(0) = 1$:

$$L(u) = \cos u = \cos(t - z)$$

which means propagation of the wave in the z direction with 1 (speed of light), that is particles at $z = z_0$will feel its effect only at $t = z_0$. In general spatial distances we measure are

$$d^2 = g_{ij}\xi^i\xi^j = (\delta_{ij} + h_{ij})\xi^i\xi^j = \delta_{ij}\Xi^i\Xi^j$$

where the latter coordinates are defined as

$$\Xi^i = \xi^i - \frac{1}{2}h^i_k\xi^k$$

and $O(h^2)$ terms are neglected. Take now a ring of (say) dust in the $z = z_0$ plane. The distance we measure after $t = 0$ are now

$$\begin{aligned}\Xi^x &= Le^{2\beta}\xi^x = e^{-2(t-z_0)}\cos(t-z_0)\xi^x \\ \Xi^y &= Le^{-2\beta}\xi^y = e^{2(t-z_0)}\cos(t-z_0)\xi^y\end{aligned}$$

Hence the ring spreads out in the x, shrinks in the y direction with an exponential rate and an additional oscillating factor.

(e) Spatial distance in the z direction is $g_{zz}(\xi^z)^2 = (\xi^z)^2$ which is a constant so the wave is in deed effects only transversal distances.

(f) This point is a straightforward calculation for a handful of nontrivial Riemann tensor components, and is left for the reader.'